The Term Structure of Credit Risk:

Estimates and Specification Tests

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In this paper we consider models of the term structure of default risk and apply them to a sample of risky Brady bonds issued by the governments of Mexico, Venezuela, and Costa Rica. The first model, which assumes that at each point in time the probability of default (given no prior default) is equal for every future period, is straightforward to implement and yields seemingly plausible estimates of default probabilities. But the dynamics of the computed default probabilities are not consistent with the model. Contrary to the model's implication that there are no anticipated changes in default probabilities, we find that the computed default probabilities do not evolve randomly. The other four models, which treat credit quality as an unobservable variable, fits the data with varying degrees of success. The first of these assumes that credit quality follows a continuous-time diffusion process and yields a closed-form solution for the default probabilities. In deriving the remaining three models we work in discrete time. Although we are unable to obtain a closed-form solution for the default probabilities, this approach affords considerable flexibility for the choice of stochastic process followed by credit quality. The results suggest that this greater flexibility is important. We find that allowing for both temporary and permanent components to the evolution of credit quality considerably reduces the estimated size of the drift parameter in the random walk and problems with the specification of the random walk model disappear.

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Valuing risky debt requires an assessment of the borrower's credit quality. A well-specified model of credit quality and its impact on the price of risky debt is an essential ingredient in examining the relative pricing of multiple debt issues from single borrower—whether the goal is to identify mispricing of existing issues or to value new issues. And when issues with different maturities or durations are considered, well-specified models of the dynamics of credit quality are needed to evaluate the relative pricing of debt instruments.

In this paper we examine alternative methods for making inferences about the value and dynamics of credit quality from market prices. In doing so we ask if the time series behavior of the prices of risky debt is consistent with five models of the time series behavior of credit risk. The first model simplifies the inference problem by assuming that at each point in time, the probability of a borrower defaulting at any future time (conditional on not having yet defaulted) is a constant.[1] Although this assumption results in attractive simplification, it implies that there is no anticipated change in the creditworthiness of the borrower. That is to say, current and anticipated future creditworthiness are assumed to be identical. That may not be the case.

In the other models, creditworthiness is treated as an unobserved variable that follows a specified stochastic process. Claessens and Pennacchi (1996) derive the first of these models by assuming credit quality follows a continuous time diffusion process.[2] They use this model to derive estimates of credit risk of Brady bonds and look at economic variables that might explain it's movements over time. Although their model provides scope for anticipated changes in credit quality and yields a closed-form expression for the bond price, it has the disadvantage of severely constraining the dynamics of credit quality. Unfortunately, it is difficult to derive closed-form solutions for stochastic processes that allow for more general credit quality dynamics.

We avoid this difficulty by moving to discrete time. Doing so has the advantage of greater flexibility because it allows us to consider virtually any stochastic process for credit quality. We consider

[1]This method, which has been used for example by Bierman and Haas (1975), Yawitz (1977), and Yawitz, Maloney, and Ederington (1985) to study the pricing of corporate debt, is used by practitioners. See, for example, Bartholomew (1991).

[2] Their model is similar to that proposed by Longstaff and Schwartz (1995).

three processes—a random walk with drift, a stationary autoregressive process, and a random walk with drift and stationary higher-frequency dynamics. This last process has the advantage of imposing fewer restrictions on the dynamics of credit quality and can therefore produce a wider range of term structures of credit risk.

We then implement the models using a sample of Brady bonds issued by the governments of Mexico, Venezuela, and Costa Rica. Although the issues we address are of concern for the pricing of risky debt in general (including corporate debt and swaps), Brady bonds make an ideal vehicle with which to examine the dynamics of credit quality. They are risky and, judging from the large fluctuations in their prices, have experienced large movements in their perceived credit risk. Moreover, the market for Brady bonds is a large and highly liquid market—accounting for less than 1/10 of outstanding international bond issues, but 1/5 of secondary market turnover in the Euroclear system. Market liquidity for the most actively-traded Latin American bonds exceeds that for the U.S. corporate and junk bonds and is comparable to all but the most liquid industrial country government bond markets [see Clark (1993, 1994)]. Brady bonds are also used as a benchmark to price new issues in what has become a booming market in debt issues.

We then subject the estimation results to specification tests. These tests show that the model that assumes the conditional probability of default to be constant is dynamically inconsistent in that changes in the computed default probabilities are predictable. The data are also clearly inconsistent with the model that assumes credit quality follows a stationary autoregressive process. The models allowing for nonstationary dynamics for credit quality provide a better fit. Of these, the model in which credit quality is assumed to exhibit stationary dynamics around a random walk provides the best fit for the bonds in our sample. The problems we find with the other specifications do not arise in this model. Moreover, it produces terms structures of credit risk that differ substantially from those implied by the other models and these differences lead to significant differences in the pricing of new bond issues.

I. The Data

Mexico, Costa Rica, and Venezuela were the first three countries to issue bonds as part of the debt and debt service reduction (DDSR) agreements. All three issued a fixed-rate and a floating-rate bond for debt conversion. The Mexican and Venezuelan fixed and floating rate issues are referred to as the par and discount bonds.[3] The Mexican bonds have the largest amount outstanding, are the most liquid, and have the smallest bid/ask spread.

The Mexican par and discount bonds were issued in March 1990 with an initial maturity of 30 years. Bank debt could be exchanged for the par bond with principal equal to the original face value of the debt. The par bond pays a fixed coupon of 6.25% of principal. Creditors could also exchange bank debt for the discount bond at a rate of $65 of principal per $100 of original face value. The discount bond pays a coupon of LIBOR plus 13/16. The principal of both bonds is guaranteed by collateral in the form of thirty-year U.S. Treasury zero-coupon bonds. A rolling interest guarantee is provided by a pool of collateral sufficient to cover eighteen months of coupon payments. Should the borrower fail to make a coupon payment, the lender will receive the coupon from the collateral.[4]

The Venezuelan par and discount bonds were issued in December 1990 with an initial maturity of thirty years. The debt conversion terms were similar to those associated with the Mexican bonds, with conversion of bank debt to discount bonds occurring at 70% of original face value and the par bond carrying a coupon rate of 6.75%. The principal guarantee and rolling interest guarantee are also similar to those attached to the Mexican bonds.[5]

The Costa Rican fixed-rate Brady bond, called the Principal Series A bond, comes with no collateral guarantee but has a rolling interest guarantee. It was issued in May 1990 with an initial maturity of 20 years, pays a coupon of 6.25% and the rolling interest guarantee covers eighteen months of interest.

We obtained daily quotes from commercial and investment banks that are active in the market and

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[3] These terms are derived from the bank debt conversion conventions used when the bonds were issued.

[4] The collateral for the discount bond is sufficient to cover eighteen months of coupons at an assumed rate of ten percent. As LIBOR is currently well below that rate, substantially more than eighteen months of coupons are guaranteed by the pool.

[5] The rolling interest guarantee covers fourteen rather than eighteen months of interest.

sample the data weekly using Wednesday prices. When Wednesday prices are not available (primarily due to holidays), the closest available price is used.

Figures 1- 3 present the prices of the Brady bonds for Mexico, Venezuela, and Costa Rica, respectively. The Mexican issues began trading at the end of March 1990 with prices around $40 for the par bond and $60 for the discount bond. Both prices rise on average over the sample, but the price of the fixed-rate bond (the Par bond) rises more rapidly that the prices of the floating-rate bonds (the Discount bond) reflecting falling long-term interest rates over the sample until 1994. In 1994, the prices of both bonds fall somewhat.[6] As interest rates rose, the price of the fixed-rate bond fell by more than the prices of the floating-rate bonds. Trading in the Venezuelan issues began in December 1990, with the discount bond priced around $70 and the par bond priced around $50. Like the Mexican issues, both prices show a sharp increase in the first quarter of 1991. Both prices fell in the second half on 1991 and throughout 1992, recovered in 1993 and then fell sharply in the first quarter of 1994. As with the Mexican bonds, the price of the fixed-rate Costa Rican Brady bond rose steadily until early 1994. After a sharp drop in March of 1994, the price of the Costa Rican bond declined over the remainder of that year.

In order to value the Brady bond components, we need prices of U.S. Treasury zero-coupon bonds. These are obtained by taking estimates of the yield curve of coupon-paying U.S. Treasury securities for bonds with three and six months and one, two, three, five, seven, ten, and thirty years to maturity. The yields are obtained from daily data reported in Barclays de Zoete Wedd's, The Rate Reference Guide to the U.S. Treasury Market. Cubic splines are then used to compute yields to maturity for each (weekly) maturity from one week to thirty years. These yields are then used recursively to compute the implied yields on zero-coupon securities and the corresponding discount factors. When valuing the floating rate bonds, we use LIBOR data obtained from the Financial Times.

II. Modeling the Term Structure of Credit Quality

At each point in time, the probability that a risky bond will default at various future dates may differ. Default probabilities may be expected to rise, fall, or vary in some non-monotonic way. What we

[6] The fall in bond prices associated with the end-1994 Mexican crisis can be seen only in the last 2 observations of the sample.

call the term structure of credit quality is a description of how, at a given point in time, default probabilities vary (if at all) across horizons.

Let $\gamma_t(s)$ be the probability at t that a payment is missed and therefore the bond is in default at t+s. The term structure of credit risk then describes how $\gamma_t(s)$ varies as a function of s. In addition, let $\rho_t(s)$ be the probability at t that default will occur at t+s conditional on no default occurring prior to t+s. This conditional probability is related to the default probabilities that describe the term structure of credit risk as, $\gamma_t(s) = \gamma_t(s-1) + \rho_t(s)(1-\gamma_t(s-1))$. Alternatively,

$$1 - \gamma_t(s) = \prod_{i=1}^{s} (1 - \rho_t(i)). \tag{1.}$$

The probability, as of t, that a bond first goes into default at t+s is,

$$\begin{aligned} \theta_t(s) &= \gamma_t(s) - \gamma_t(s-1) \\ &= \rho_t(s) \prod_{i=1}^{s-1} (1 - \rho_t(i)). \end{aligned} \tag{2.}$$

We consider the value of the principal, the interest payments, and the interest guarantees separately. The principal payment for the Mexican and Venezuelan Brady bonds is guaranteed by U.S. Treasury zero coupon bonds. The present value of the principal is therefore simply the face value of the bond discounted at the appropriate U.S. zero-coupon rate. Valuing the interest payments is less straightforward, however, since they are risky. If the default risk associated with holding the bond is completely diversifiable or, alternatively, investors are risk neutral, the value of a coupon payment can be computed by discounting the expected payment (contractual payment multiplied by the probability that the payment will be made) at a rate given by the yield on a U.S. Treasury zero-coupon bond maturing when the coupon payment is to be made. Alternatively, we may view the probabilities in (1) and (2) as risk neutral probabilities. Under this interpretation, the same valuation method for the coupon payments allows for the presence of non-diversifiable default risk or risk averse investors.

We first consider the coupon payments on a fixed-rate bond and then turn to a floating-rate bond. Let F be the face value of the bond and let r be the coupon rate and suppose that the bond is not in default at time t. If the bond was riskless, the time-t value of a coupon due at t+s would be $rF/(1+r_{t,s})^s = rFp_t(s)$

where $r_{t,s}$ is the yield and $p_t(s)$ is the price at t of a U.S. Treasury zero-coupon bond maturing at t+s. Since there is a non-zero probability of default, to obtain the expected present value of the payment we must multiply by $1-\gamma_t(s)$, the probability that the bond is not in default at t+s.

Next, suppose that a pool of collateral is used to guarantee q of the bond's coupon payments. If the bond goes into default at t+s, the coupon payments at t+s, t+s+1, and so on will be met by drawing on the pool of collateral until the pool is exhausted. Once the pool has been depleted, no further coupon payments will be made.[7] The value at t of a pool first used at t+s is $rF[p_t(s)+p_t(s+1)+\cdots+p_t(s+q)]$. Again to obtain the expected present (time-t) value of the pool, we need to multiply by $\theta_t(s)$, the probability that default first occurs at t+s.

The price of the bond, P_t, should be equal to the expected present value of the individual components of the bond. Adding the expected present values of the coupons, the rolling interest guarantee, and the principal (which has value $Fp_t(N)$, where t+N is the maturity date), we obtain,

$$P_t = rF\sum_{i=1}^{N}\{p_t(i)(1-\gamma_t(i)) + \theta_t(i)\sum_{j=1}^{\min(q,N-i)} p_t(i+j)\} + p_t(N)F \cdot \qquad (3.)$$

The value of a floating rate bond can be obtained in similar way and used to estimate the default probability. The only difference in valuation arises from the adjustment of the interest rate each period. Let $r_t(i)F$ be the coupon to be paid at time t+i on a floating rate bond with face value F. The coupon rate $r_t(i) = L_t(i) + sp$, where $L_t(i)$ is the one-period LIBO rate determined at t+i-1 for payment at t+i and sp is the spread over LIBOR that is paid on the bond. Of course future LIBOR is unknown and cannot be used to value the bond. Since the market value of $L_t(i)$ paid at t+i is the discounted value of $rf_t(i)$, where $rf_t(i)$ is the forward rate, we use the implied forward one-period interest rate computed from the term structure of zero coupon rates.[8] The implied one-period U.S. Treasury rate to be paid at t+i is given by $p_t(i-1)/p_t(i)-1$.

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[7] The collateral may not cover all of the q coupon payments. If this is the case, the last payment will be partial.

[8] Using the implied forward rates does not require that we adopt an expectations theory of the term structure as the following investment strategy makes clear. Consider a floating rate bond that pays r_{t+1} in period t+2. The period t market value of this payment can be obtained by considering the following strategy. Let f be the period t forward price of a one period bond issued in t+1. (a) Buy $(1-f)/(1+r_t)$ units of a one period bond at t; (b) sell one forward contract; (c) with the proceeds from the first two parts, buy $r_{t+1}/(1+r_{t+1})$ units of a one period bond issued at t+1. The cash flows are $(1-f)/(1+r_t)$ in period t, zero in period t+1, and r_{t+1} in period t+1. The market value of

U.S. Treasury rates are lower than LIBOR so we add a forecast of the spread of LIBOR over one-period Treasury rates to obtain the implied forward one-period LIBOR. We then have $rf_t(i) = p_t(i-1)/p_t(i) - 1 + d_t(i)$, where $d_t(i)$ is the forecast at t of spread of LIBOR over one-period Treasury rates at t+i.[9] Equation (3) then becomes,

$$P_t = F\sum_{i=1}^{N}\{r_t(i)p_t(i)(1-\gamma_t(i)) + \theta_t(i)\sum_{j=1}^{\min(q,N-i)}[r_t(i+j)p_t(i+j)\} + p_t(N)F. \quad (4.)$$

III. A Geometric Term Structure of Credit Risk

The first method we use to value the risky bonds assumes that $\rho_t(s) = \rho_t$. At each point in time, the conditional probability is assumed to be constant at each horizon, s, but the conditional probability can change over time. The term structure of credit risk, which is described by $\gamma_t(s) = 1 - (1-\rho_t)^s$, is monotonic and $\gamma_t(s)$ approaches one as the horizon increases. Although assuming that conditional default probabilities are constant at each point in time yields a simple and tractable means of making inferences about credit quality from observed bond prices, it imposes strong restrictions on the behavior of perceived credit quality. In particular, the assumption does not allow for anticipated improvements or declines in creditworthiness - current and expected future creditworthiness are assumed to be identical.

Estimates of ρ_t can be obtained by solving (3) and (4) for the value of ρ_t that equates the market price with the expected present value of the payments. These estimates are plotted in figures 4 through 6. The estimated probabilities for the two Mexican bonds, found in figure 4, point to a substantial improvement in credit quality in the first half of 1991. Credit quality is relatively stable for the remainder of the sample except for a significant improvement beginning in late 1993 around the time that the U.S. Congress approved the NAFTA and ending late in the first quarter of 1994 around the time of the assassination of presidential candidate Colosio, and sharp increase at the end of 1994 around the time of the peso crisis.

As can be seen in figure 5, the estimated default probabilities for the Venezuelan bonds also

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r_{t+1} paid in t+2 must then be $(1-f)/(1+r_t)$. Since $f = 1/(1+rf_{t+1})$, this market value is $rf_{t+1}/(1+r_t)(1+rf_{t+1}) = rf_{t+1}\, p_t(2)$, which is the present value of rf_{t+1}.

[9] The forecast is formed by estimating the first-order autoregression $d_t = \alpha + \beta d_{t-1} + v_t$. The forecast, $d_t(i) = \alpha(1-\beta^i)/(1-\beta) + \beta^i d_t$.

indicate that perceived credit quality improved in the first half of 1991. After fluctuating for about 18 months credit quality fell around the first quarter of 1993 and then fell abruptly in the first quarter of 1994. Figure 6 contains a plot of the estimated default probabilities for the Costa Rican bond, which bear a surprising resemblance to those obtained for the Mexican bonds. Perceived credit quality rises in the first half of 1991 and fluctuates for the next few years before rising again in late 1993 and early 1994 and then falling in March of 1994 and again around the time of the Mexican peso crisis.

Is the assumption of constant conditional default probabilities consistent with the data? If the assumption is correct, the estimated probabilities should evolve randomly. If, on the other hand, it is possible to predict changes in the default probabilities, the assumption that current and expected future creditworthiness are identical is inconsistent with the data. Table 1 contains the results of a runs test, which is a simple nonparametric test of the hypothesis that the probabilities evolve randomly. A run occurs when two or more successive observations in a series are increasing or decreasing. The number of runs in repeated samples of size T is distributed asymptotically as a normal random variable with mean (2T-1)/3 and variance (16T-29)/90.[10] The table reports the number of runs that occur in each of the estimated default probability series along with a standard normal statistic to test the null hypothesis that each series is random. As can be seen from the table, we reject the null hypothesis at any reasonable significance level for all five bonds because there are too few runs in the sample to be consistent with randomness. Thus the data appear to be inconsistent with assumption that there is no anticipated change in credit quality. We now turn to a series of models of the evolution of credit quality.

IV. Models of Credit Quality as an Unobserved Variable

We consider models in which the evolution over time of creditworthiness determines the evolution of the default probabilities. Suppose that the underlying credit quality of the borrower at some time t can be represented as the realization of an unobservable random variable, x_t. If the x falls to zero at some time t+s, we assume that the borrower defaults on the payments that are due at t+s. The four models that we

[10] Kennedy and Gentle (1980), p. 171.

consider in this section differ in the stochastic process followed by x_t. The first model assumes that x_t is a continuous-time diffusion process and that the bond is in default if x falls to zero regardless of whether a payment is due on that date. This model yields a closed-form solution for the probabilities $\gamma_t(s)$ and $\theta_t(s)$ and therefore for the bond price. Although this model allows for anticipated changes in credit quality, it severely restricts those dynamics. In addition, the assumption that default can occur when no payment is due is problematic. We move to discrete time to derive the other three models and assume that default can only occur when a payment is due. Although we cannot derive closed-form solutions for the probabilities and bond prices, the absence of closed-form solutions presents no problems for estimation. And we gain considerable flexibility in the choice of stochastic process.

IV.1. The Claessens-Pennacchi Model

Model 1: Claessens and Pennacchi (1996) assume that x_t evolves as a simple Brownian motion,

$$dx_t = \mu dt = \sigma dz_t \tag{5.}$$

where dz is a standard Wiener process and treat default as an absorbing barrier at zero. The drift parameter, μ, allows one to distinguish between current and future creditworthiness. A positive value of μ indicates that creditworthiness is expected to increase while a negative value of μ indicates anticipated declines in creditworthiness. When the drift parameter is zero, creditworthiness evolves randomly.

Given the assumption that of credit quality follows a diffusion process, Claessens and Pennacchi derive both the default probabilities and the values of the bonds using standard results for the distribution of x_t with an absorbing barrier at zero and its corresponding first passage distribution.[11] Suppose the bond is not currently in default, $x_t > 0$. The probability that the issuer misses a payment and the bond is therefore in default at t+s is the probability that $x_n \le 0$ for some $n \in (t+s-1, t+s)$. This probability is,

$$\gamma_t(s) = (\text{Prob}(x_{t+s-1} \le 0 \mid x_t) + \text{Prob}(x_{t+s-1} > 0)\,\text{Prob}(x_n \le 0, n\,\varepsilon\,(t+s-1, t+s) \mid x_{t+s-1} > 0) \tag{6.}$$

[11] See for example Cox and Miller (1982) or Ingersoll (1987).

Unless a payment is due every period, the Claessens-Pennacchi model therefore assumes that default can occur even is no payment is due between t+s-1 and t+s. Using the results from Geske (1979), we can rewrite (6) as

$$\gamma_t(s) \;=\; N\left(\frac{-x_t-\mu(s-1)}{\sigma\sqrt{s-1}}\right) + N_2\left(\frac{x_t+\mu(s-1)}{\sigma\sqrt{s-1}}, \frac{-x_t-\mu s}{\sigma\sqrt{s}}, -\frac{\sqrt{s-1}}{\sqrt{s}}\right) \tag{7.}$$

$$+\; e^{\frac{-2\mu x_t}{\sigma^2}} N_2\left(\frac{x_t-\mu(s-1)}{\sigma\sqrt{s-1}}, \frac{-x_t+\mu s}{\sigma\sqrt{s}}, -\frac{\sqrt{s-1}}{\sqrt{s}}\right)$$

where $N(\cdot)$ is the standard normal distribution function and $N_2(\cdot)$ is the bivariate normal distribution function. The probability that the rolling interest guarantee is used at t+s is then,

$$\theta_t(s) \;=\; N\left(\frac{-x_t-\mu s}{\sigma\sqrt{s}}\right) + e^{\frac{-2\mu x_t}{\sigma^2}} N\left(\frac{-x_t+\mu s}{\sigma\sqrt{s}}\right) \tag{8.}$$

$$-\left(N\left(\frac{-x_t-\mu(s-1)}{\sigma\sqrt{s-1}}\right) + e^{\frac{-2\mu x_t}{\sigma^2}} N\left(\frac{-x_t-\mu(s-1)}{\sigma\sqrt{s-1}}\right)\right)$$

We can then use these probabilities to value the components of the bonds as in (3) and (4) and equate the value of the components to the market price of the bond.

IV.2. Discrete Time Models

Suppose that the K payments on a bond are due at t+m and then every n periods thereafter. If an issuer fails to make the τth payment, the bond is in default and the random variable $x_{t+m+(\tau-1)n} \leq 0$.[12] The probability as of t that the bond issuer will default on the τth payment, conditional on no prior default is,

[12] Unlike Claessens and Pennacchi, we work explicitly in discrete time rather than with a discrete time version of a continuous-time model. We therefore consider the probability that $x \leq 0$ only on date on which a payment is due.

$$\rho_t(m + (\tau - 1)n) = \Pr(x_{t+m+(\tau-1)n} \le 0)\prod_{i=1}^{\tau-1}\Pr(x_{t+m+(\tau-1)n} > 0)$$
$$= \Pr(x_{t+m+(\tau-1)n} \le 0)\prod_{i=1}^{\tau-1}\left[1 - \Pr(x_{t+m+(\tau-1)n} \le 0)\right]. \tag{9.}$$

Once we specify a stochastic process for x, we can compute these probabilities and use (1) and (2) to obtain the probabilities of default at any point in time, $\gamma_t(s)$, and first default at any point in time, $\theta_t(s)$. This approach has the advantage of greater flexibility because we are not confined to distributions that yield closed-form solutions for the probabilities $\gamma_t(s)$ and $\theta_t(s)$.

Model 2: The first of the discrete-time models we consider assumes that credit quality is a random walk with drift.

$$x_t = \mu + x_{t-1} + \eta_t \qquad \eta_t \sim N(0,\sigma^2). \tag{10.}$$

This model therefore differs from the Claessens-Pennacchi model in the way default is modeled but not in the assumed stochastic process followed by credit quality. The probability as of t that a bond is in default when the τth payment is due is given by the product in (1) along with (9) and,

$$\Pr(x_{t+m+(i-1)n} \le 0) = \frac{1}{\sqrt{[m + (i-1)n]}\sigma} N\left(\frac{-[m + (i - 1)n]\mu - x_t}{\sqrt{[m + (i-1)n]}\sigma}\right). \tag{11.}$$

Similarly, the probability of first default (and therefore first use of the rolling interest guarantee) is given by (2) and (9). The prices of fixed and floating rate bonds can then be computed as in (3) and (4).

Model 3: As with the Claessens-Pennacchi model, this first discrete-time model severely restricts the dynamics of credit quality. The next model we examine assumes that credit quality exhibits mean-reverting dynamics. In particular we assume that credit quality follows stationary autoregressive process,

$$x_t - x_{t-1} = k(\bar{x} - x_{t-1}) + \eta_t \qquad \eta_t \sim N(0,\sigma^2), \tag{12.}$$

where the rate of reversion to mean $\bar{x}$ is given by k.

The probability that the bond is in default is again given by (1), (9), and the probability,

$$\Pr(x_{t+m+(i-1)n} \leq 0) = \frac{1}{\Sigma(m+(i-1)n)} \, N\left(\frac{-\bar{x} - (x_t - \bar{x})(1-k)^{m+(i-1)n}}{\Sigma(m+(i-1)n)} \right), \tag{13.}$$

where
$$\Sigma(j)^2 = \frac{\sigma^2}{(1-k)^2}\left[1 - (1-k)^{2j}\right].$$

The probability of first default is then again given by (2) and the prices of fixed and floating rate bonds can be expressed as the expected present value of the payments as in (3) and (4).

Model 4: The final model we consider allows for both stationary and nonstationary components in the dynamics of credit quality. In particular, we assume that credit quality follows an ARIMA(1,1,1) process.

$$\Delta x_t = (1-\alpha)\mu + \alpha \Delta x_{t-1} + \eta_t - \lambda \eta_{t-1} \qquad \eta_t \sim N(0,\sigma^2) \tag{14.}$$

As with the previous two models, the conditional and unconditional default probabilities are readily derived form the probability that the continuous random variable, x, is less than zero on some future payment date. In appendix II we show how to derive that probability in a way that readily generalizes to other stochastic processes.

V. Estimation and Specification Testing

Each of the models described in section IV can be combined with data on bond prices and interest rates to obtain estimates of the key parameters. Each yields an expression in which the probabilities and therefore the bond values are functions of x_t/σ. We normalize σ=1 and write,

$$P_t = G(x_t) + \epsilon_t, \tag{15.}$$

where the function $G(\cdot)$ is given in (3) and (4) and ϵ_t is a random measurement error. We assume ϵ_t is distributed as $N(0,\sigma_\epsilon^2)$. The equations of motion for x_t for the three discrete-time models are given above and for the Claessens-Pennacchi model we work with a discrete version of the diffusion process.

$$x_t = x_{t-\Delta t} + \mu\Delta t + \eta_t \qquad \eta_t \sim N(0,\Delta t). \tag{16.}$$

For each model we now have a two-equation system involving the unobservable x_t written in state space form. The measurement equation, which relates the unobservable state variable, x_t, to observable prices, is (15). The evolution of the state variable is given by (10), (12), (14), or (16). In appendix 1, we describe how a Kalman filter may be used to produce maximum-likelihood estimates of the parameters of each process and of the unobservable variable x_t.

Table 2 contains the maximum likelihood estimates of the four models for each of the five bonds. The first two models differ in their treatment of default—in the first default occurs when x first reaches zero even if no payment is due on that date while in the second default can occur only when $x \leq 0$ and a payment is due. The first row of results for these two models contains estimates of the drift parameter and its asymptotic standard error. The estimated drifts are all small but highly significant. Four of the five estimated drifts from the first model are negative, indicating a decline in anticipated credit quality. Two of these four estimates change sign, however, when the model with a discrete-time random walk is estimate. More problematic, the signs on the estimated drifts for the par and discount bonds differ for the Venezuelan bonds in the first model and the Mexican bonds in the second model. Differing signs suggest a problem with the model's specification—credit quality for two issues from the same borrower should be similar. The second row contains estimates of the standard deviation of the measurement error. The estimates from both of the first two models are small and insignificantly different from zero for all five bonds, indicating that the model fits the data exceptionally well.[13]

The table also contains the results of two diagnostic tests of the specification. Both diagnostics examine the innovations in the filter's estimates of the unobservable state variable representing credit quality. The first diagnostic tests for serial correlation in the innovations. The second diagnostic examines whether the filter adequately captures the heteroskedasticity in the state variable by looking for serial

[13] One cannot use a t ratio formally to evaluate whether the error is significant or not because the regularity conditions are not satisfied. When the measurement error variance is constrained to zero we obtain results essentially identical to those reported here.

correlation in the squared innovations. The probability values are presented along with the test statistics. There is some evidence of autocorrelation in the residuals for the Venezuelan bonds (model 2). The test for heteroskedasticity also points to specification problems for the Venezuelan discount bond (model 2) and the Costa Rican bond (model 1).

The estimates for model 3, which assumes that credit quality exhibits stationary autoregressive dynamics, suggest that reversion to the mean is extremely slow. The estimated rates of convergence are 0.005 or less for the Mexican and Venezuelan bonds and 0.01 for the Costa Rican bond. The estimates for the Mexican bonds suggest that the mean of the credit quality variable is about 1.25 standard deviations above zero. The estimate for the Costa Rican bond is similar. The estimated means for the Venezuelan bonds are a puzzle, one estimate is small and statistically insignificant while the other is large (about 2.7 standard deviations above zero) and highly significant. Again, these differing estimates suggest specification problems. The diagnostics also point to significant specification problems for the Mexican and Venezuelan bonds. There is substantial evidence of residual autocorrelation and heteroskedasticity for all four of these bonds. The model with stationary autoregressive dynamics appears to fit the Costa Rican data well.

The final set of estimates in table 2 are for the model in which credit quality is assumed to exhibit stationary dynamics around a random walk with drift. This model clearly has the best overall fit. Adding stationary dynamics to the random walk with drift reduces the size of the estimated drift substantially. In addition, there is no sign of the problems evident in the first two models where the signs of the estimated drifts differed for bonds issued by the same borrower.[14] There appears to be both an autoregressive and a moving average component to the stationary dynamics and the autoregressive component generally appears to be fairly persistent. The diagnostics also suggest that this model fits the data well. None of the bonds show evidence of residual autocorrelation at standard significance levels and only the estimates for the two Mexican bonds appear to show signs of heteroskedasticity.

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[14]Although there is quite a large difference between the AR parameter estimates for the Mexican bonds, the default probabilities implied by these estimates are quite similar. Because the AR and MA parameters in the Discount model imply the presence of a common factor in the process for Δx, beyond very short horizons the default probabilities implied by both models are similar to the probabilities implied by a random walk process with drift.

The differences in the models can yield significant differences in the term structures of credit quality that they imply. Figure 7 depicts the term structures of credit quality—the survival probabilities, 1 - $\gamma_t(s)$, for values of s up to 30 years—implied by three of the models we consider on one date (January 5, 1994). As expected, the term structures computed from the constant conditional probability model show a gradually declining survival probability for each of the five bonds.

In addition, we compute the survival probabilities for the Claessens-Pennacchi model and the ARIMA(1,1,1) model using the estimated parameters from those models along with estimated value of the state variable, x_t, for January 5, 1994. As is clear from figure 7, the ARIMA model yields a greater variety of term structure shapes than does either of the other two models. The differences for the two Venezuelan bonds are the most striking. The estimated drift for Venezuelan par bond obtained from the Claessens-Pennacchi model is positive. Because credit quality is anticipated to improve, the current value of the state variable estimated to be fairly low. As a result the survival probability is quite low initially and then rises substantially over time. The ARIMA model yields a term structure that captures key features that appear in the other two models. Both models that allow for some credit quality dynamics point to higher long-run credit quality than does the constant conditional probability model. But the ARIMA model is able to do so without the dramatic reduction in near-term credit quality the comes out of the Claessens-Pennacchi model.

The differences in the term structures can lead to substantial differences in the valuation of new debt instruments. The Claessens-Pennacchi model undervalues a 30-year straight bond with a coupon rate of 6.5% and face value of $100 by about $3.50 relative to the ARIMA model using the term structure computed from the estimates for the Venezuelan par bond. The relative undervaluation is even greater (about $11.50) when we use the term structure estimated from the estimates for the Venezuelan discount bond. These valuation differences indicate the differences in the models are economically meaningful.

V. Concluding Remarks

We have considered several models of the term structure of default risk and then applied them to a sample of risky Brady bonds issued by the governments of Mexico, Venezuela, and Costa Rica as part of debt and debt service reduction deals. The first model, which assumes that at each point in time the probability of default (given no prior default) is equal for every future period, is straightforward to implement and yields seemingly plausible estimates of default probabilities. But the dynamics of the computed default probabilities are not consistent with the model. Contrary to the model's implication that there are no anticipated changes in default probabilities, we find that the computed default probabilities do not evolve randomly.

The other four models, which treat credit quality as an unobservable variable, fits the data with varying degrees of success. The first of these, the Claessens-Pennacchi model, assumes that credit quality follows a continuous-time diffusion process and yields a closed-form solution for the default probabilities. Although their model allows for anticipated dynamics for credit quality, it severely limits those dynamics. In deriving the remaining three models we work in discrete time. Although we are unable to obtain a closed-form solution for the default probabilities, this approach affords considerable flexibility for the choice of stochastic process followed by credit quality. The results suggest that this greater flexibility is important. We find that allowing for both temporary and permanent components to the evolution of credit quality is important. When we allow for stationary dynamics around a random walk with drift we find that the parameters of both parts of the process are statistically significant. More importantly, we find that allowing for stationary dynamics considerably reduces the estimated size of the drift parameter in the random walk and problems with the specification of the random walk model disappear.

Appendix 1

This appendix describes the algorithm used to estimate the unobservable component model of section III.2. The filtering results presented below are discussed in Harvey (1990).

The model to be estimated comprises equation (15):

$$P_t = G(x_t) + \epsilon_t \qquad \epsilon_t \sim N(0,\sigma_\epsilon^2)$$

and the equation of motion for one of the processes for x_t shown in (10), (12), (14) or (16). For convenience, we write these processes in companion form,

$$x_t = C z_t$$
$$z_t = A z_{t-1} + u_t \qquad \Omega = E(u_t u_t')$$

The state space form of all the models can now be written as

$$P_t = V(z_t) + \varepsilon_t \qquad \varepsilon_t \sim N(0,\sigma_\varepsilon^2)$$
$$z_t = A z_{t-1} + u_t \qquad \Omega = E(u_t u_t')$$

where $V(z_t) = G(Cz_t)$.

Estimates of μ and σ_ϵ^2 are obtained by maximizing the likelihood function for this model constructed with the use of the non-linear Kalman Filter. The filtering equations are

$$P_t = V(z_{t/t-1}) + u_t \tag{A1}$$

$$R_t = \nabla V(z_{t/t-1})\Sigma_{t/t-1} \nabla V(z_{t/t-1})' + \sigma_\epsilon^2 \tag{A2}$$

$$z_{t+1/t} = A(x_{t/t-1} + k_{t/t-1}u_t) \tag{A3}$$

$$\Sigma_{t+1/t} = A\,[1 - k_{t/t-1} \nabla V(z_{t/t-1})]\Sigma_{t/t-1} A' + \Omega \tag{A4}$$

$$k_{t/t-1} = \Sigma_{t/t-1} \nabla V(z_{t/t-1})/R_t^{-1} \tag{A5}$$

where $z_{t/t-1}$ is the estimate of z_t given information at t-1, $\nabla V(z_{t/t-1})$ is the vector gradient of V(.) evaluated at $z_{t/t-1}$, and $\Sigma_{t/t-1}$ is the variance of z_t given information available at t-1. Notice that the filter effectively replaces $V(z_t)$ with the linear approximation $V(z_{t/t-1}) + \nabla V(z_{t/t-1})(z_{t/t-1})[z_t - z_{t/t-1}]$.

Estimates of the parameters in the x_t process and σ_ϵ^2 are found by maximizing the likelihood

$$\sum_{t=1}^{T} - \log(2\pi) - \log(R_t) - \frac{u_t^2}{2\ R_t}$$

where u_t and R_t are derived from (A1) - (A5).

Since V(.) is a very complicated non-linear function, the gradients V'(.), are calculated numerically. To initialize the filter at t=1, we set $\Sigma_{1/0}$ equal to zero, and $z_{1/0}$ equal to z_0, a parameter to be estimated. The likelihood function was maximized with the Downhill Simplex algorithm that does not require the computation of gradients of the likelihood function.

Once the maximum likelihood estimates are obtained, the filter is used to calculate the estimates of z_t using data up to time t:

$$z_{t/t} = z_{t/t-1} + k_{t/t-1} u_t. \tag{A6}$$

Appendix II

The probability that $x \leq 0$ at the future payment date t+m+(I-1)n is,

$$\Pr(x_{t+m+(i-1)n} \leq 0) = \frac{1}{\sqrt{C\Sigma_{t+m+(i-1)n|t}C^{/}}} N\left(\frac{Cz_{t+m+(i-1)n|t}}{\sqrt{C\Sigma_{t+m+(i-1)n|t}C^{/}}}\right),$$

$\Sigma_{t+m+(i-1)n|t}C$

where $Cz_{t+m+(i-1)n|t}$ is the expected value of $x_{t+m+(I-1)n}$ and $C\Sigma_{t+m+(i-1)n|t}C^{/}$ the variance both conditioned on on time t data.To obtain these means and variances for the ARIMA(1,1,1) model we write in companion form as shown above:

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \eta_t \\ 1 \end{bmatrix} = \begin{bmatrix} (1+\alpha) & -\alpha & -\lambda & (1-\alpha)\mu \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_{t-2} \\ \eta_{t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \eta_t$$

where C is a row vector with one as its first element and zeros elsewhere so that it selects the first element of the vector z. Th exoected value of $x_{t+m+(I-1)n}$ can then be computed as $CA^{m+(I-1)n}z_t$ and $C\Sigma_{t+m+(i-1)n|t}C^{/}$ can be computed from the recursion,

$$\Sigma_{t+m+(i-1)n|t} = A\,\Sigma_{t+m+(i-1)n-1|t}\,A' + \Omega \qquad .$$

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Table 1

Runs Tests

Bond	Actual # Runs	Expected # Runs	Test Statistic
Mexico Par	126	167	-6.34
Mexico Discount	114	167	-8.26
Venezuela Par	104	140	-6.39
Venezuela Discount	104	140	-6.39
Costa Rica Principal A	109	146	-6.31

Notes: A run occurs when two or more successive observations are increasing or decreasing. The test statistic is distributed asymptotically as standard normal under the null hypothesis that the conditional probabilities evolve randomly.

Table 2					
		Mexico			
		Par		Discount	
Model 1	μ	-0.020	(0.002)	-0.038	(0.002)
	$\sigma_\varepsilon(\times 10)$	0.010	(0.010)	0.003	(0.008)
	$\chi^2(I)$	5.177	(0.521)	7.531	(0.274)
	$\chi^2(II)$	8.421	(0.209)	6.999	(0.321)
Model 2	μ	0.009	(0.001)	-0.023	(0.001)
	$\sigma_\varepsilon(\times 10)$	0.002	(0.009)	0.001	(0.006)
	$\chi^2(I)$	4.774	(0.573)	7.678	(0.263)
	$\chi^2(II)$	7.427	(0.283)	5.373	(0.497)
Model 3	k	0.004	(0.001)	0.005	(<0.001)
	$\bar{x}$	1.348	(0.045)	1.135	(0.058)
	$\sigma_\varepsilon(\times 10)$	7.243	(0.449)	6.465	(0.438)
	$\chi^2(I)$	33.279	(<0.001)	31.961	(<0.001)
	$\chi^2(II)$	104.642	(<0.001)	157.643	(<0.001)
Model 4	μ	-0.001	(<0.001)	-0.001	(0.001)
	α	0.141	(0.013)	0.973	(0.004)
	λ	1.042	(0.002)	1.005	(<0.001)
	$\sigma_\varepsilon(\times 10)$	0.005	(0.010)	0.035	(0.005)
	$\chi^2(I)$	6.547	(0.365)	10.173	(0.117)
	$\chi^2(II)$	16.625	(0.011)	10.873	(0.092)

Table 2 cont

		Venezuela				Costa Rica	
		Par		Discount			
Model 1	μ	0.007	(0.001)	-0.004	(0.008)	-0.114	(0.001)
	$\sigma_\varepsilon(\times 10)$	0.001	(0.017)	0.002	(0.024)	0.005	(0.001)
	$\chi^2(I)$	2.887	(0.823)	9.041	(0.171)	6.760	(0.343)
	$\chi^2(II)$	3.615	(0.729)	1.615	(0.952)	46.986	(<0.001)
Model 2	μ	0.007	(<0.001)	0.027	(0.003)	-0.067	(0.003)
	$\sigma_\varepsilon(\times 10)$	<0.001	(0.006)	0.001	(0.043)	<0.001	(0.012)
	$\chi^2(I)$	53.638	(<0.001)	47.476	(<0.001)	4.809	(0.582)
	$\chi^2(II)$	4.779	(0.573)	14.468	(0.025)	6.558	(0.364)
Model 3	k	0.004	(<0.001)	0.002	(0.001)	0.010	(0.001)
	$\bar{x}$	0.076	(0.266)	2.687	(<0.001)	1.281	(0.122)
	$\sigma_\varepsilon(\times 10)$	4.937	(0.005)	4.194	(<0.001)	3.904	(0.770)
	$\chi^2(I)$	44.064	(<0.001)	83.746	(<0.001)	5.805	(0.445)
	$\chi^2(II)$	143.971	(<0.001)	1.861	(0.932)	2.390	(0.881)
Model 4	μ	0.003	(0.001)	0.006	(0.001)	-0.021	(0.004)
	α	0.717	(0.087)	0.822	(0.073)	0.789	(0.002)
	λ	1.017	(0.004)	1.010	(0.003)	0.945	(0.056)
	$\sigma_\varepsilon(\times 10)$	0.001	(0.010)	0.003	(0.018)	0.019	(0.023)
	$\chi^2(I)$	10.513	(0.105)	8.854	(0.182)	3.734	(0.713)
	$\chi^2(II)$	1.987	(0.921)	5.555	(0.475)	7.901	(0.245)

Notes: The estimated models have the form:

$$P_t = G(x_t) + \varepsilon_t \qquad \varepsilon_t \sim N(0,\sigma_\varepsilon^2)$$

where V(.) is the pricing function given in (xx), and x_t is the unobserved state variable that follows one of the following processes:

$$x_t = x_{t-\Delta t} + \mu\Delta t + \eta_t \qquad \text{Model 1}$$

$$x_t = x_{t-1} + \mu + \eta_t \qquad \text{Model 2}$$

$$\Delta x_t = k(\bar{x} - x_{t-1}) + \eta_t \qquad \text{Model 3}$$

$$\Delta x_t = (1-\alpha)\mu + \alpha\Delta x_{t-1} + \eta_t - \lambda\eta_{t-1} \qquad \text{Model 4}$$

where $\eta_t \sim N(0,1)$.

Figure 1: Mexican Brady Bond Prices

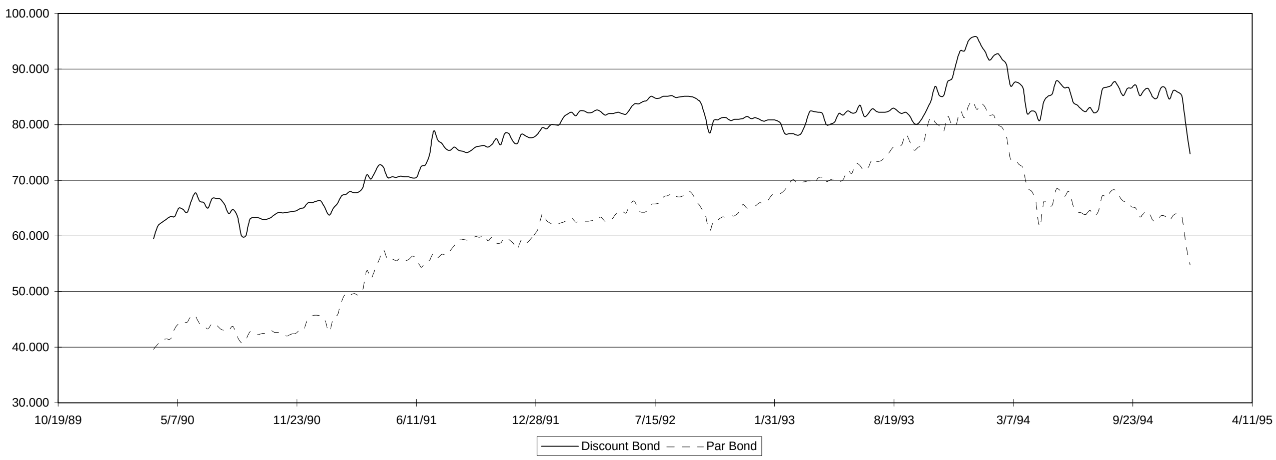


Figure 2: Venezuelan Brady Bond Prices

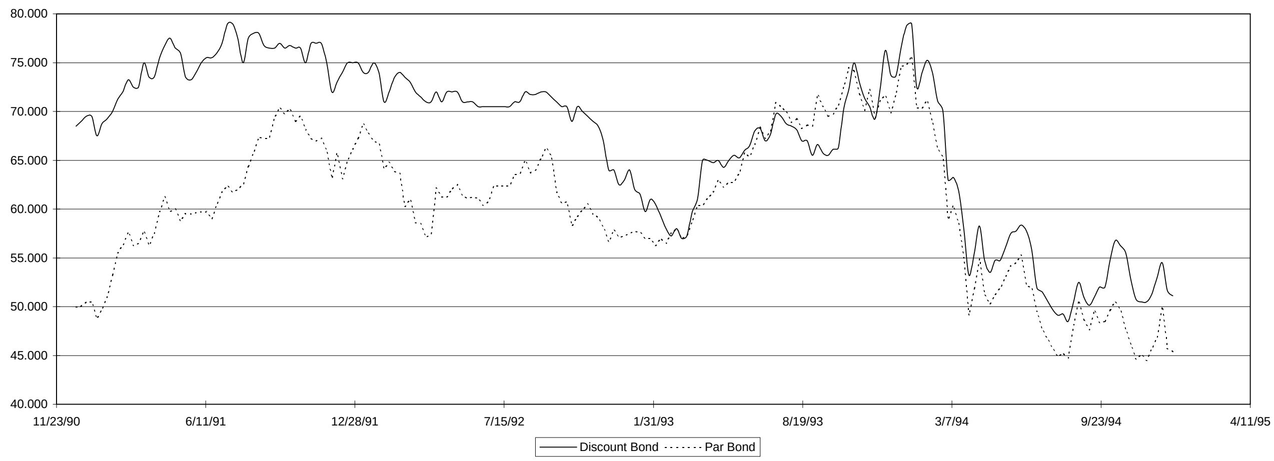


Figure 3: Costa Rican Brady Bond Prices

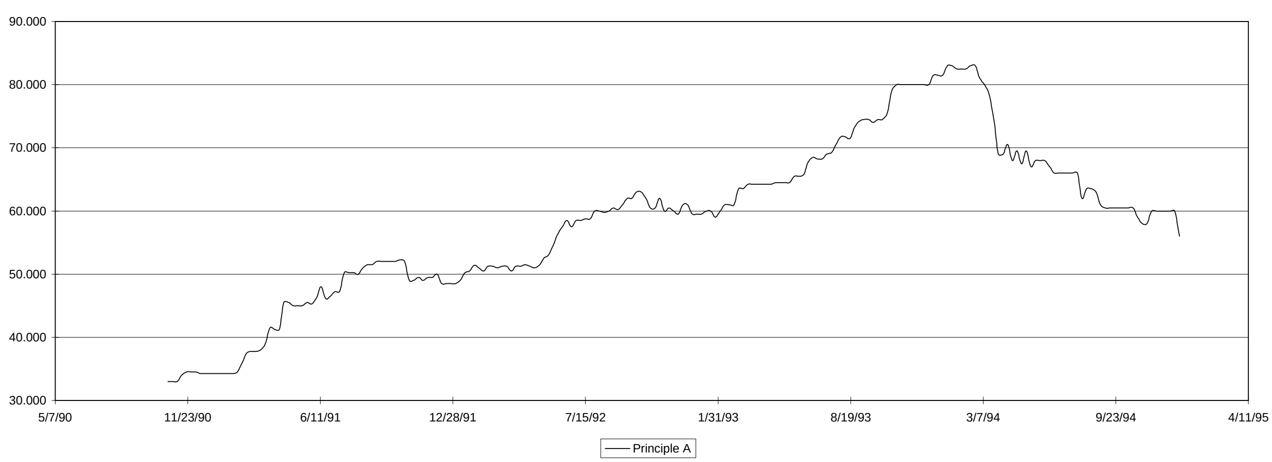


Figure 4: Mexican Default Probabilities

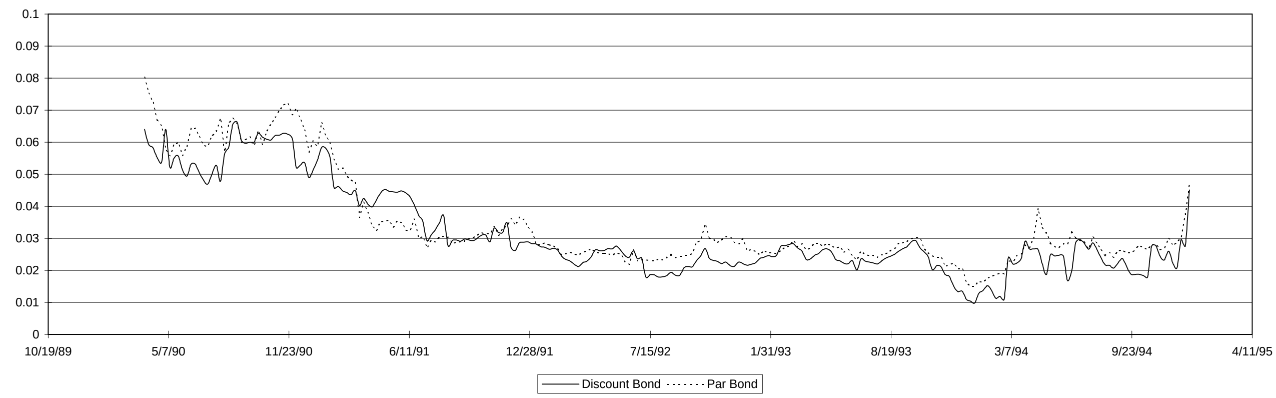


Figure 5: Venezuelan Default Probabilities

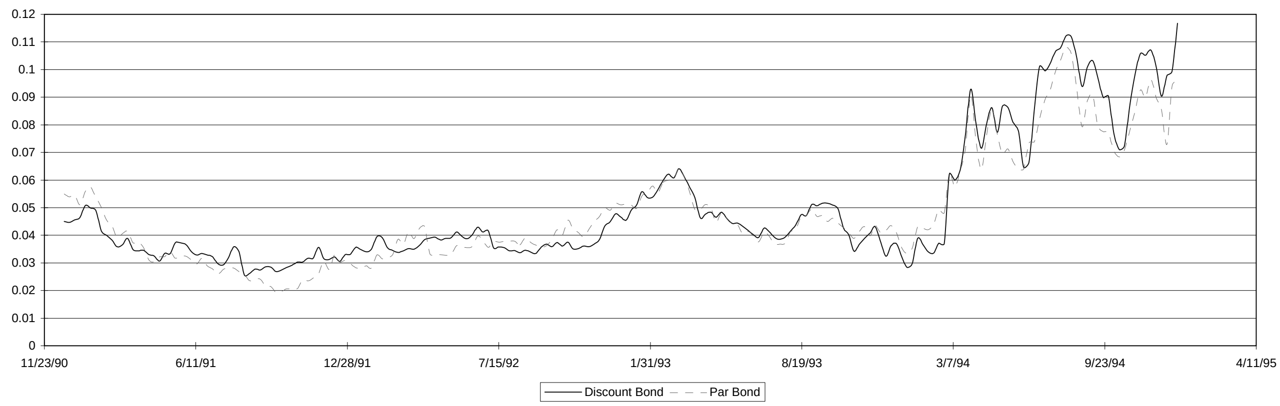


Figure 6: Costa Rican Default Probabilities

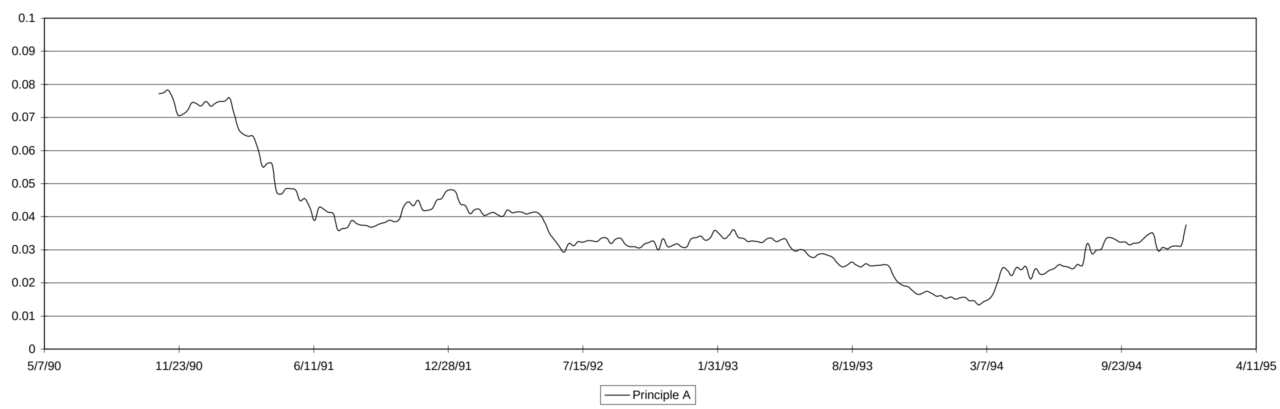
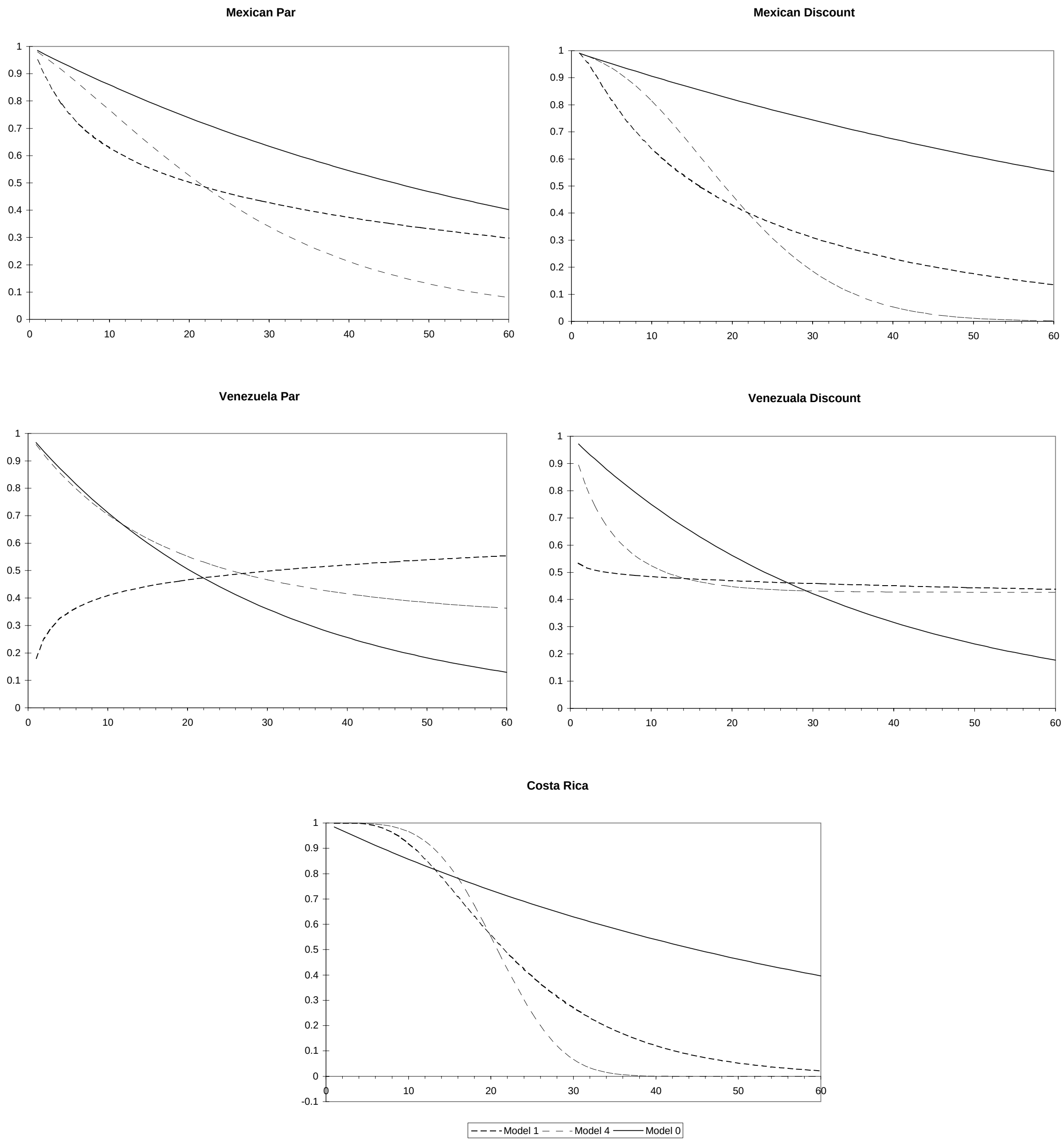


Figure 7



Mexican Par
Mexican Discount
Venezuela Par
Venezuala Discount
Costa Rica
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-0.1
0 10 20 30 40 50 60
Model 1
Model 4
Model 0