Chapter 3: Direct Proofs

§ 3.1: Getting Started on a Problem

Let's review some steps for studying the truth of a statement.

Step 1, Examples: If possible, look at some specific examples, including extreme cases, to help understand if a statement is true and **why** it might be true. This step doesn't always help, but should always be considered. When working examples, make sure you trying to understand what each clause in the statement means, why each clause is there, if the statement seems true, and, most importantly, **why** the statement is true. THINK ABOUT THE EXAMPLES.

Step 2, Assumed: Carefully list what is given and/or assumed, possibly using definitions so that what is given is in a usable form. This often involves the choose method. IF YOU DO THIS STEP PROPERLY, MANY PROBLEMS WILL ALMOST SOLVE THEMSELVES.

Step 3, To Show: Carefully describe what must be shown, again using definitions to rewrite what must be shown in a usable form. IF YOU DON'T KNOW WHAT YOU NEED TO PROVE, YOU WON'T BE ABLE TO PROVE IT.

In Steps 2 and 3, take careful note of the phrases "for all" and "there exists" to help identify what variables are (from the choose method) assumed as given, and what variables are considered unknown and are to be found. Sometimes the phrase "for all" is implied. For example, if the statement begins "Let $x > 1$" or "Suppose $x > 1$", what the statement is really saying is "For all $x > 1$."

In each of the next several examples, the goal is to see how each of the three steps contribute to understanding the problem and finding a proof. The actual proof is of secondary importance. In the next section, we will begin focusing on developing proofs.

Example 3.1: Let $n, m \in \mathbb{Z}$. If n and m are both odd or n and m are both even, then $n + m$ is even.

Step 1, Examples: We should add several pairs of odd integers, including cases where both are negative and only one is negative. In all cases, the sum will be even. We should do the same thing for pairs of even integers. Think about why the examples work. When adding even numbers, each number has a factor of 2, so 2 can be factored out of the sum. When adding odd integers, each integer is a multiple of 2, plus an extra 1, so when added together, we get an extra 2, making the sum even.

Step 2, Assumed: The statement is of the form $p \vee q \Rightarrow r$ where $p \equiv \{n, m \in \mathbb{O}\}$, $q \equiv \{n, m \in \mathbb{E}\}$, and $r \equiv \{n + m \in \mathbb{E}\}$.

Using the definitions of even and odd, we can assume there exists two integers, $k, l \in \mathbb{Z}$ such that

$$p \equiv \{n = 2k \text{ and } m = 2l\}$$

or $$q \equiv \{n = 2k + 1 \text{ and } m = 2l + 1\}$$

So we know one of these two statements must be true, and could then do cases, Case 1 where p is assumed true and Case 2 where q is assumed true.

Step 3, To Show: We must show $n + m$ is even. Using the definition, we must show

$$r \equiv \{\exists\, a \in \mathbb{Z} \ni n + m = 2a\}$$

From Step 2, if we assume p is true, then the variables k and l are treated as if they were known integers. To show r is true, we treat the variable a as if it were an unknown which we have to find. Note that

$$n + m = 2k + 2l = 2(k + l)$$

so we can let $a = k + l$ and we have solved for the unknown variable in terms of the known variables. This occurs quite frequently. The reader should prove the second case in Example 3.1 by showing $q \Rightarrow r$.

Example 3.2: Show for every integer n, $n^3 + 3n$ is even.

Step 1, Examples: Let's try several values for n to see what is happening. We organize our examples in Table 3.1.

Table 3.1: Check of claim

n	n^3	$3n$	$n^3 + 3n$
1	1	3	4
2	8	6	14
3	27	9	36
4	64	12	76
5	125	15	140
6	216	18	234

The claim seems to be true; the numbers in the last column are all even. By breaking the sum into parts, though, we notice a pattern. When n is even, we get the sum of two even integers. When n is odd, we get the sum of two odd integers. So the even total comes about in different ways, depending on whether n is even or odd. This suggests cases,

$$\{n \text{ is an integer}\} \equiv \{n \text{ is an even integer}\} \vee \{n \text{ is an odd integer}\}$$

We can now write what is assumed in a usable form.

Step 2, Assumed: Originally, it didn't look like we could assume much, but the examples suggest we break the problem into the two cases where n is even and when n is odd. Using the definitions, we assume there exists an integer k such that

$$n = 2k \text{ or } n = 2k + 1$$

Step 3, To Show: $n^3 + 3n$ is even, which by the definition means we must show there exists an integer j such that $n^3 + 3n = 2j$.

As in Example 3.1, the integer k from Step 2 is treated as a known and the integer j from Step 3 is an unknown integer that will be solved for in terms of k. In the proof of the claim in Example 3.2, if $n = 2k$, then

$$n^3 + 3n = (2k)^3 + 3(2k) = 2(4k^3 + 3k)$$

and $j = 4k^3 + 3k$. On the other hand, if $n = 2k + 1$, then

$$n^3 + 3n = (2k{+}1)^3 + 3(2k{+}1) = 8k^3 + 12k^2 + 12k + 4 = 2(4k^3 + 6k^2 + 6k + 2)$$

and $j = 4k^3 + 6k^2 + 6k + 2$.

We could have simplified the proof for Example 3.2 by applying Example 3.1. For example, assuming that $n = 2k$, we get that n^3 is even and $3n$ is even, so by Example 3.1, the sum is even. Alternatively, if $n = 2k + 1$, we can show n^3 is odd and $3n$ is odd, so again by Example 3.1, the sum is even.

The statement in Example 3.2 could have been proven a third way by writing

$$n^3 + 3n = n(n^2 + 3)$$

and considering two columns in a Table similar to Table 3.1, in which one column is the n-column and another column is the $n^2 + 3$ column. In this case, when n is even, one factor is even, and when n is odd, the other factor is even. The proof is similar to the one given here.

Example 3.3: Show that if $|x| > 10$, then $x^2 + 40 > 14x$.

Step 1, Examples: If we substitute values such as $x = 10.1,\ 100,\ -10.1,\ -200$ into the inequality, we will see that the values satisfy the inequality. It is not clear from the examples why this works. Sometimes it helps to try values not assumed, that is x-values such that $|x| \leq 10$. Some values will also satisfy the inequality and some won't. The key is trying $x = 10$, in which case $x^2 + 40 = 14x$. This seems to be the key.

Step 2, Assumed: $p \equiv \{|x| > 10\}$

Note that what is assumed is not in a usable enough form. Absolute values are often difficult to work with, algebraically. We will use cases to rewrite it in more usable form. If

$$p \equiv \{|x| > 10\},\ p_+ \equiv \{x > 10\} \text{ and } p_- \equiv \{x < -10\}$$

then $$p \equiv p_+ \vee p_-$$

We can rewrite what is assumed as $x > 10$ or $x < -10$. A usable way to rewrite $x > 10$ is

$$x = 10 + a,\ a \in \mathbb{R}^+$$

Similarly, we could rewrite $x < \ -10$ as

$$x = -10 - a,\ a \in \mathbb{R}^+$$

Step 3, To Show: $q \equiv \{x^2 + 40 > 14x\}$ or

$$q \equiv \{x^2 - 14x + 40 > 0\}$$

When dealing with inequalities, it is often helpful to have all of the terms on the same side.

The proof for Example 3.3 is relatively easy, once Steps 2 and 3 have been done. Substitution of $x = 10 + a$ into $x^2 - 14x + 40$ gives, after simplification,

$$6a + a^2$$

which is clearly positive since $a \in \mathbb{R}^+$. A similar result holds in the second case, where $x = \ -10 - a$.

Example 3.4: Let f and g be two real-valued functions with domain $D \subseteq \mathbb{R}$. Show that if f and g are bounded, then $h = f + g$ is also bounded.

Step 1, Examples: We might begin with some real simple examples, such as $f(x) = 3$ and $g(x) = 2$. Clearly, f is bounded by 3 and g is bounded by 2. The function $h(x) = f(x) + g(x) = 5$ is bounded by 5. We think we see what is happening; the bound for h is the sum of the bounds for f and g. If we try one more simple example, say $f(x) = 3$ and $g(x) = -2$, then again, the bound for f is 3 and the bound for g is 2 (remember, we are bounding $|g(x)|$). But the lowest bound for $h(x) = f(x) + g(x) = 1$ is 1, not the sum of the bounds, 5. On the other hand, 5 is still a bound for h, and no one said we had to get the best bound.

As a second example, let $D = \{0, 1\}$ and let

$$f(x) = \begin{cases} 5 & x = 1 \\ 2 & x = 1 \end{cases} \text{ and } g(x) = \begin{cases} 4 & x = 1 \\ 3 & x = 0 \end{cases}$$

The lowest bound for f is 5 and the lowest bound for g is 4. We see that the lowest bound for

$$h(x) = f(x) + g(x) = \begin{cases} 9 & x = 1 \\ 5 & x = 0 \end{cases}$$

is 9, the sum of the two bounds. On the other hand, if

$$f(x) = \begin{cases} 5 & x = 1 \\ 2 & x = 1 \end{cases} \text{ and } g(x) = \begin{cases} 3 & x = 1 \\ 4 & x = 0 \end{cases}$$

the bounds for f and g are the same, but the lowest bound for $f+g$ is 8, not 9. But 9 is still a bound for $f+g$, and we were only looking for a bound.

Note: When constructing examples, it is usually best to begin with as simple an example as possible. Once you think you understand the problem, try your conjectures on progressively more complicated examples.

Step 2, Assumed: We go back to the definition, and assume that

$$\exists\, M \in \mathbb{R}^+ \ni \forall\, x \in D,\ |f(x)| \leq M$$
$$\exists\, N \in \mathbb{R}^+ \ni \forall\, x \in D,\ |g(x)| \leq N$$

From this point on, we can assume that M and N are known numbers, and the inequalities hold for every x in the domain.

Step 3, To Show: Again, from the definition, we must show that

$$\exists\, K \in \mathbb{R}^+ \ni \forall\, x \in D,\ |h(x)| = |f(x) + g(x)| \leq K$$

The number K is presently unknown, and must be found.

The actual proof of the statement in Example 3.4 just uses the ideas from the examples, letting

$$K = M + N$$

Now K exists, and is "known." We then apply the triangle inequality from Problem 2.31, and the two conditions from Step 2 to get that for every $x \in D$,

$$|h(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq M + N = K$$

so we are finished.

Note that while the conditions in Step 2 and Step 3 of Example 3.4 are similar since they use the definition of bounded, the ways they are used in the proof are quite different. In particular, the "there exists M and N" allowed us to treat them as if they were known, while "show there exists K" implied K was unknown and had to be found.

Example 3.5: Let $i, j \in \mathbb{Z}$. Show that if $i \underset{7}{=} 2$ and $j \underset{7}{=} 5$, then $ij \underset{7}{=} 3$. (Recall from Problem 2.20 that $n \underset{k}{=} m$, read as "n equals m mod k" means that k divides $n - m$.)

Step 1, Examples: We need values for i and j such that 7 divides $i-2$ and $j-5$. Some values that work for i are $i = 9$, $i = -5$, and $i = 30$. Some values that work for j are $j = 12$, $j = -2$, and $j = 47$. The claim is that $ij - 3$ is divisible by 7. We see that

$$\begin{aligned}
&\text{if } i = 9 \text{ and } j = 12 \text{ then } ij - 3 = 9(12) - 3 = 105 = 7(15)\\
&\text{if } i = -5 \text{ and } j = 47 \text{ then } ij - 3 = -5(47) - 3 = -238 = 7(-34)\\
&\text{if } i = 30 \text{ and } j = 47 \text{ then } ij - 3 = 30(47) - 3 = 1407 = 7(201)
\end{aligned}$$

so the claim appears to be true, although the examples haven't helped in understanding why. Let's consider one of the examples in a little more detail, using the facts that $i = 7k + 2$ and $j = 7l + 5$

$$30(12) = [7 \cdot 4 + 2][7 + 5] = 7^2 \cdot 4 + 7 \cdot 2 + 7 \cdot 20 + 10$$

Since $10 = 7 + 3$, we see that each term has a 7, with a 3 left over, so the product is of the form

$$ij = 7k + 3$$

which is what we wanted to show. This last example seems to be the key. It seems a 3 is always left over after all the multiples of 7 have been combined.

Note: Sometimes you need to go into more depth in the example to understand the process.

Step 2, Assumed: Using the definition of "mod", we have assumed that

$$\begin{aligned}
&\exists\, k \in \mathbb{Z} \ni i = 7k + 2\\
&\exists\, l \in \mathbb{Z} \ni j = 7l + 5
\end{aligned}$$

At this point, k and l can be treated as known integers.

Step 3, To Show: Again, using the definition, we must show that

$$\exists\, n \in \mathbb{Z} \ni ij = 7n + 3$$

Remember that n is unknown and must be found. Often, we can write the unknown in terms of the "knowns" given in the Step 2.

The proof for Example 3.5 is trivial from here. We substitute for i and j from step 2, and after a little algebra, get that

$$ij = (7k + 2)(7l + 5) = 7(7kl + 5 + 2) + 10 = 7(7kl + 7) + 7 + 3 = 7(7kl + 8) + 3$$

We now let $n = 7kl + 8$, and are finished since $ij = 7n + 3$.

Examples 3.4 and 3.5 had similar conditions in both Steps 2 (assumed) and Step 3 (to show), but the variables were treated quite differently in the two steps. As in Examples 3.1 and 3.2, the variables introduced in Step 2, M and N in Example 3.4, and k and l in Example 3.5, were treated as known quantities, while the variables introduced in Step 3, K in Example 3.4 and n in Example 3.5, were unknowns that needed to be found. In both examples, the unknowns were eventually written in terms of the "known" quantities from Step 2,

$K = M + N$ in Example 3.4, and $n = 7kl + 8$ in Example 3.5

§ 3.2: Forward/Backward Proofs

Once we have worked some examples so that we understand the problem, and have carefully written what we are assuming and what we have to prove, we are ready to begin trying to construct the proof. This usually consists of two steps, 1) Thinking about the problem and constructing an informal proof, and 2) Constructing a formal proof from the informal proof in which we are careful with all of the details. These steps are analogous to writing a rough draft for a paper, and writing a final draft. Often, there will be several revisions before we get from our informal proof to a clear, well-written formal proof.

To show that an implication of the form $p \Rightarrow q$ is true, it is simplest to try a forward approach, replacing statement p with other statements that follow from it, until we reach statement q, that is, $p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \ldots \Rightarrow q$. Unfortunately, forward proofs are often difficult to find. A variation on the forward proof is the forward/backward proof. In this case, we proceed from both ends and work toward the middle, that is, we begin with $p \Rightarrow p_1 \Rightarrow \ldots$ and go as far as we can. Then we start with q and go as far backwards as we can, $q \Leftarrow q_1 \Leftarrow \ldots$. Hopefully, at some point, we arrive at a common point. From this, we can then rewrite the steps as a **forward** proof. This indicates two additional steps in verifying statements.

Step 4, Thinking: From Steps 1-3 in Section 3.1, we think through why the statement is true. For now, this means we use a forward/backward approach until we have constructed an informal proof.

Step 5, Formal Proof: Using the ideas in Step 4, we construct a formal, forward proof, being careful that all details have been taken care of.

Following are several examples in which the forwards/backwards approach is used. In each of these examples, Steps 1-5 are described in detail. Until you become proficient in constructing proofs, you should break the problem into these steps.

Example 3.6, Forward/Backward Proof: Show that if $x > 11$, then

$$x^2 > 9x + 22$$

Step 1, Examples: If $x = 12$, then $x^2 = 144 > 9x + 22 = 130$. We should consider some extreme cases. If $x = 11.01$, then

$$x^2 = 121.2201 > 9x + 22 = 121.09$$

On the other hand, if $x = 100$, then

$$x^2 = 10000 > 9x + 22 = 922$$

It is not clear what is happening yet. We might wonder why the condition, $x > 11$, is important. So we try $x = 11$ and get that

$$x^2 = 121 = 9x + 22$$

so $x = 11$ seems to be a root of $x^2 - 9x - 22$.

Step 2, Assumed: Our statement can be rewritten as

$$\forall\, x > 11,\ x^2 > 9x + 22$$

Using the choose method, we "choose" $x > 11$. We could consider our given as

$$p \equiv \{x > 11\}$$

We now consider x as a known number greater than 11.

Step 3, To Show: We want to show that

$$q \equiv \{x^2 > 9x + 22\} \equiv \{x^2 - 9x - 22 > 0\}$$

Step 4, Thinking: If we start with $x > 11$, it is difficult to see what we should do next. In this case, we begin with q and rewrite it as

$$x^2 - 9x - 22 > 0$$

We factor this equation, giving $(x - 11)(x + 2) > 0$. We now see that if $x > 11$, then each factor is positive, so the product is positive. We now **understand** the proof, but the proof is not clearly written. The simplest proof is to begin with p and arrive at q, as in the next step.

Step 5, Formal Proof: Assume that $x > 11$ and show that $x^2 > 9x + 22$.

Proof: Choose $x > 11$ (Choose Method). Since $x > 11$, then

$$(x - 11) > 0$$

Since $x + 2$ is positive for $x > 11$, we can multiply both sides of our inequality by $x + 2$ without changing the inequality sign, giving

$$(x - 11)(x + 2) > 0$$

Expanding gives $x^2 - 9x - 22 > 0$ or

$$x^2 > 9x + 22\square$$

The proof in Example 3.6 could have been done in a different, but equally good manner. We could let $x = 11 + a$ where $a > 0$. Substitution into $x^2 - 9x - 22$ gives $a^2 + 13a > 0$ since $a > 0$. Often there are numerous ways of constructing a proof. It can be instructive to see how different people approach a problem.

It is acceptable to work backwards when thinking out a proof as long as the finished proof goes from p to q. One reason to do this is to insure all of our steps are valid. The forward proof often hides the thinking that went into developing the proof. If we showed someone the finished proof of Example 3.6, they would have no idea how that proof was developed. This is often the case in mathematics. We struggle for hours on different parts of a proof, then when we organize it and write it as a forward proof, as in Example 3.6, the proof seems obvious if not almost trivial. Don't let that bother you.

In the next example, we are going to investigate the boundaries of a set in $\mathbb{R}$. First, we will develop an understanding of what the boundary of a set is.

If we have an open interval,

$$(a, b) = \{x : a < x < b\}$$

then our intuition tells us that the boundary points for (a, b) are the end-points of the interval,

$$B = \{a, b\}$$

This is also the boundary of the closed interval

$$[a, b] = \{x : a \leq x \leq b\}$$

Note that for the open interval (a, b), the boundary points were not in the set, but for the closed interval $[a, b]$, the boundary points were in the set.

If we have a finite collection of points, say

$$S = \{1, 2, 3\}$$

then we would consider each of these points to be a boundary of S, so the set S equals the set of boundary points for S. But what would we consider as the boundary of a set, such as

$$S = \{x : x = \frac{1}{n},\ n \in \mathbb{Z}^+\}$$

Each of the points in this set would seem to be a boundary point, but since the points are all "bounded" by 0, would $b = 0$ also be considered a boundary point? Our intuition should say, yes.

Intuitively, a boundary point seems to separate the set S from the set S^c. This leads to the following definition.

Definition: Let $S \subseteq \mathbb{R}$. The number $b \in \mathbb{R}$ is a **boundary value** for S if and only if

whenever $b \in (x, y)$, then $(x, y) \cap S \neq \emptyset$ and $(x, y) \cap S^c \neq \emptyset$

We call the set

$$B_S = \{b \in \mathbb{R} : b \text{ is a boundary value for } S\}$$

the **boundary** of S.

To summarize, b is a boundary value if

$$q \equiv \{\langle \forall\ (x, y),\ b \in (x, y) \rangle,\ \langle S^c \cap (x, y) \neq \emptyset \rangle \wedge \langle S \cap (x, y) \neq \emptyset \rangle\}$$

that is, every interval containing b also contains points in both S and S^c.

Now let's discuss what it means to not be a boundary value. This means b is not a boundary value if $\neg q$ is true. We can write this as

$$\begin{aligned}\neg q \quad &\equiv \neg\{\langle\forall\ (x,y),\ b\in(x,y)\rangle,\ \langle S^c\cap(x,y)\neq\emptyset\rangle\wedge\langle S\cap(x,y)\neq\emptyset\rangle\}\\ &\equiv \{\neg\langle\forall\ (x,y),\ b\in(x,y)\rangle,\ \langle S^c\cap(x,y)\neq\emptyset\rangle\wedge\langle S\cap(x,y)\neq\emptyset\rangle\}\\ &\equiv \{\langle\exists\ (x,y),\ b\in(x,y)\rangle,\ \neg\Big\langle\langle S^c\cap(x,y)\neq\emptyset\rangle\wedge\langle S\cap(x,y)\neq\emptyset\rangle\Big\rangle\}\end{aligned}$$

We remember that

$$\neg(p\wedge r)=\neg p\vee\neg r$$

Also note that

$$\neg\langle S^c\cap(x,y)\neq\emptyset\rangle=S^c\cap(x,y)=\emptyset$$

So we can summarize

$$\neg q\equiv\{\langle\exists\ (x,y),\ b\in(x,y)\rangle,\langle S^c\cap(x,y)=\emptyset\rangle\vee\langle S\cap(x,y)=\emptyset\rangle\Big\rangle\}$$

To put this another way, b is not a boundary value if there exists an interval containing b that is contained entirely in S or S^c.

To show b is a boundary value, we must show something for every interval (x,y) containing b. To show b is not a boundary value, we must find one interval (x,y) that satisfies a certain condition. Again, notice the careful use of $\exists$ and $\forall$.

In the next example, we show how to use the definition for boundary value.

Example 3.7: Let's see how the definition for boundary implies that if $S=(0,1)$, then the boundary is $B=\{0,1\}$. This consists of two parts. First, we must show that $b=0$ and $b=1$ are boundary values. But we must also show that every other real number is not a boundary value.

Part 1: We will show that $b=0$ is a boundary value.

Step 1: We must show that every interval containing 0 also contains points in the interval $(0,1)$ and points not in the interval, that is, points in

$$S^c=\{x : x\leq 0 \text{ or } x\geq 1\}$$

The interval $(x,y)=(-0.6,0.2)$ contains -0.3 and $0.1,$ the first being in S^c and the second being in S. Similarly, the interval $(x,y)=(-1000,2000)$ contains -500 and 0.5, again, We construct some examples, the first being in S^c and the second being in S. It seems like we could also use half the lower bound, $x/2,$ as a point in S^c, or we could just use 0 as the point in S^c. Notice we used a different process for finding a point in S. When the upper bound, $y,$ for the interval was less than 1, we could use $y/2$ as the point in S. When $y>1$, we could just use 0.5 as the point in S.

Step 2: We have assumed that

$$S = (0, 1)$$

and using the choose method, we are assuming we have an arbitrary interval (x, y) containing 0. The usable form of this is

$$x < 0 < y$$

Remember that x and y can be treated as known numbers.

Step 3: We have to show that

$$S^c \cap (x, y) \neq \emptyset \text{ and } S \cap (x, y) \neq \emptyset$$

The usable form of this statement is that we have to show there exists two numbers, a and b such that

$$a \in S^c \cap (x, y) \text{ and } b \in (0, 1) \cap (x, y)$$

The numbers a and b are treated as unknown numbers we must find. We can give them in terms of x and y.

Step 4: Let's first think about the value for a. We know $x < 0 < y$ and we want $x < a < y$. For $a \in S^c$, we must also have that $a \leq 0$ or $a \geq 1$. We could let $a = 0$ or $a = x/2$ (or $a = x/n$ for any $n \in \mathbb{R}^+$). There are lots of choices for a, but we only need one.

Now let's think about b. We need

$$0 < b < 1 \text{ and } x < b < y$$

Here we have a problem. We could try $b = 0.5$ which satisfies the first condition, but not necessarily the second condition, since y might be less than 0.5. On the other hand, we could try $b = y/2$, which satisfies the second condition, but not necessarily the first, since y might be larger than 2. This problem is easily solved by letting

$$b = \min\{0.5, y/2\}$$

Since we know y, we can always find b.

Step 5: We now construct the formal proof. Let $S = (0, 1)$. Then

$$S^c = (-\infty, 0] \cup [1, \infty)$$

Choose an arbitrary interval, (x, y) such that $0 \in (x, y)$. We know $x < 0 < y$. Let $a = 0$. Since $a \in (x, y)$ and $a \in S^c$, then

$$a \in (x, y) \cap S^c$$

Let $b = \min\{0.5, y/2\}$. Then we know that

$$b \leq 0.5 \text{ and } b \leq y/2$$

We also know that $b > 0$ since $y > 0$. Therefore

$$b \in (x, y)$$

since $x < 0 < b \leq y/2 < y$. Also,

$$b \in (0, 1)$$

since $0 < b \leq 0.5 < 1$. Therefore

$$b \in (x, y) \cap S$$

We have now shown the conditions for 0 being a boundary value.□

The proof that $b = 1$ is a boundary value is similar and will be left to the reader.

Note: When a proof has several similar parts, it is common practice to prove one part and then just state that the other parts can be shown in a similar manner. You must be careful that this is actually true.

Part 2: We now must show that if $b \neq 0$ and $b \neq 1$, then b is not a boundary value.

Step 1: Suppose $b = -1$. We must find an interval (x, y) that contains -1 and which is entirely within S or S^c. Since b is in S^c, then

$$(x, y) \subseteq S^c = \{x : x \leq 0 \text{ or } x \geq 1\}$$

This is easy. Let $(x, y) = (-2, 0)$. There are lots of other equally valid intervals.

Suppose instead that $b = 0.5$. Then $b \in S$ so $(x, y) \subseteq S = (0, 1)$. This again is easy. Just let $(x, y) = (0, 1)$.

Let's try one more extreme value, say $b = 1.0001$. Since $b \in S^c$, then $(x, y) \subseteq S^c$. The only trick is that $x \geq 1$ and $x < 1.0001$. There are lots of choices, but we let $(x, y) = (1, 2)$.

Step 2: We assume that b is not 0 or 1.

Step 3: We must find (x, y) such that

$$\langle b \in (x, y) \rangle \wedge \left\langle\ (x, y) \subseteq S \vee (x, y) \subseteq S^c \right\rangle$$

Step 4: As we saw in the examples, how we choose (x, y) depends on whether $b \in S$ or $b \in S^c$. There seem to be 3 natural cases,

$$b < 0,\ 0 < b < 1,\ b > 1.$$

If $b < 0$, then we need to find $(x, y) \subseteq S^c$, that is, find x and y such that

$$x < b < y \leq 0$$

(We have just rewritten what needs to be shown in a usable form considering the particular case we are trying.) One possibility is $(x, y) = (b - 1, 0) \subseteq S^c$.

The case in which $b > 1$ is similar, $(x, y) = (1, b + 1)$.

The case in which $0 < b < 1$ is the simplest of all, $(x, y) = (0, 1)$.

Step 5: We now prove that if $b \neq 0$ and $b \neq 1$, then b is not a boundary value. We consider 3 cases.

Case 1: Assume $b < 0$. Let $(x, y) = (b - 1, 0)$. Then

$$b \in (b-1, 0) \text{ and } (b - 1, 0) \subseteq S^c = (-\infty, 0] \cup [1, \infty)$$

so b is not a boundary value.

Case 2: Assume $0 < b < 1$.. Let $(x, y) = (0, 1)$. Then

$$b \in (0, 1) \text{ and } (0, 1) \subseteq S = (0, 1)$$

so b is not a boundary value.

Case 3: Assume $b > 1$. Let $(x, y) = (1, b + 1)$. Then

$$b \in (1, b + 1) \text{ and } (1, b + 1) \subseteq S^c = (-\infty, 0] \cup [1, \infty)$$

so b is not a boundary value.

We have now shown that if $S = (0, 1)$, then $B = \{0, 1\}$.□

It is acceptable to work backwards when thinking out a proof as long as the finished proof goes from p to q. One reason to do this is to insure all of our steps are valid. The forward proof often hides the thinking that went into developing the proof. If we showed someone the finished proof of Example 3.2, they would have no idea how that proof was developed. This is often the case in mathematics. We struggle for hours on different parts of a proof, then when we organize it and write it as a forward proof, as in Example 3.2, the proof seems obvious if not almost trivial. Don't let that bother you.

We might recall from earlier courses that functions such as $f(x) = x,\ f(x) = x^3$, and $f(x) = x^5$ are symmetric about the origin. Another way of stating this is that these functions have rotational symmetry with a 180° rotation. Such functions are called **odd** functions. The trigonometric function $f(x) = \sin(x)$ is also symmetric about the origin, and is considered odd. Similarly, functions such as $f(x) = x^2,\ f(x) = x^4$ and $f(x) = x^6$ are symmetric about the y-axis. Such functions are considered **even** functions. The trigonometric function $f(x) = \cos(x)$ is also symmetric about the y-axis, and is considered even. The precise definition for even and odd is:

Definition: The function f is **even** if $\forall\, x \in D$, $f(x) = f(-x)$. The function is **odd** if $\forall\, x \in D$, $f(x) = -f(-x)$.

Note that the definition does not use the geometric property of even and odd, but the equivalent algebraic property. This would allow the definition to be extended to other dimensions where we couldn't use our geometric intuition.

It is clear that every polynomial can be written as the sum of an even function and an odd function. The even function would just be the sum of the even powers in the polynomial and the odd function would just be the sum of the odd powers. In the next example, we consider what may be a surprising result; that any function with a symmetric domain, no matter how strange, can be written as the sum of an even function and an odd function.

Example 3.8: For this example, assume $D \subseteq \mathbb{R}$ such that if $x \in D$, then $-x \in D$.

The goal of this problem is to show that for any function f with domain D, there exists an even function g and an odd function h such that $\forall\, x \in D$,

$$f(x) = g(x) + h(x)$$

Step 1, Examples: For a complicated problem, such as this seems to be, we begin with a very simple example. Let $D = \{-1{,}1\}$ and

$$f(x) = \begin{cases} 11 & x = 1 \\ 2 & x = -1 \end{cases}$$

We need to find an even function g and an odd function h such that $f(x) = g(x) + h(x)$ for every $x \in D = \{-1{,}1\}$. The domain for g and h is also $D = \{-1{,}1\}$. Therefore, we have four unknowns we need to find, $g(1)$, $g(-1)$, $h(1)$, and $h(-1)$.

Since g must be even, we must have that

$$g(1) = g(-1) \tag{3.1}$$

Similarly, h is odd, so

$$h(1) = -h(-1). \tag{3.2}$$

We also want that

$$\begin{aligned} f(1) = 11 = g(1) + h(1) \\ f(-1) = 2 = g(-1) + h(-1) \end{aligned}$$

We now use equations (3.1) and (3.2) to get that

$$2 = g(1) - h(1)$$

We now add the equations $11 = g(1) + h(1)$ and $2 = g(1) - h(1)$ to get that

$$13 = 2g(1) \text{ or } g(1) = 6.5$$

Substitution into the other equations gives that

$$g(-1) = 6.5,\ h(1) = 4.5, \text{ and } h(-1) = -4.5.$$

We now see that the functions

$$g(x) = 6.5,\ x \in \{1, -1\}, \text{ and } h(x) = \begin{cases} 4.5 & x = 1 \\ -4.5 & x = -1 \end{cases}$$

satisfy the desired conditions, that is, g is even, h is odd, and $f = g + h$.

But this was just a simple example. Let's try a slightly more complicated example.

Suppose

$$f(x) = \begin{cases} 10 & x = 2 \\ 3 & x = 1 \\ 15 & x = 0 \\ -7 & x = -1 \\ 4 & x = -2 \end{cases}$$

We want to find an even function g and an odd function h such that $f(x) = g(x) + h(x)$ for every $x \in D = \{-2, -1, 0, 1, 2\}$. So we want the following conditions to be satisfied:

$$\begin{aligned} &f(x) = g(x) + h(x),\ x \in \{-2, -1, 0, 1, 2\} \\ &g(2) = g(-2),\ g(1) = g(-1),\ g(0) = g(-0) \\ &h(2) = -h(-2),\ h(1) = h(-1),\ h(0) = -h(-0) \end{aligned}$$

The condition that $h(0) = -h(-0) = -h(0)$ gives that $2h(0) = 0,$ so

$$h(0) = 0 \text{ and so } f(0) = g(0) = 15$$

As in the previous example, we solve the 2 equations

$$\begin{aligned} &f(2) = 10 = g(2) + h(2) \\ &f(-2) = 4 = g(-2) + h(-2) = g(2) - h(2) \end{aligned}$$

to get that

$$g(2) = 7 \text{ and } h(2) = 3, \text{ so that } g(-2) = 7 \text{ and } h(-2) = -3.$$

Similarly, we get that

$$g(1) = -2 \text{ and } h(1) = 5, \text{ so that } g(-1) = -2 \text{ and } h(-1) = -5.$$

So if

$$g(x)=\begin{cases}7 & x=2\\ -2 & x=1\\ 15 & x=0\\ -2 & x=-1\\ 7 & x=-2\end{cases} \text{ and } h(x)=\begin{cases}3 & x=2\\ 5 & x=1\\ 0 & x=0\\ -5 & x=-1\\ -3 & x=-2\end{cases}$$

then we see that g is even, h is odd, and $f = g + h\,\forall\, x \in D$.

Step 2, Assumed: Let f be a function with domain D. We cannot assume much at this point.

Step 3: To Show: We need to find functions g and h such that, for every $x \in D$,

$f(x) = g(x) + h(x)$ (which also means $f(-x) = g(-x) + h(-x)$)
$g(x) = g(-x)$
$h(x) = -h(-x)$

Step 4: Thinking: We now think of x as a fixed value. We have four unknown numbers to find, $g(x)$, $g(-x)$, $h(x)$, and $h(-x)$. This means we need 4 equations. So we will work backwards from the desired equations that we want to show. In those equations, $f(x)$ and $f(-x)$ are treated as known numbers, so we actually have 4 equations. The four equations that need to be satisfied are

$$\begin{aligned}&f(x) = g(x) + h(x)\\ &f(-x) = g(-x) + h(-x)\\ &g(x) = g(-x)\\ &h(x) = -h(-x)\end{aligned}$$

We make the substitutions $g(x) = g(-x)$ and $-h(x) = h(-x)$ in the equation $f(-x) = g(-x) + h(-x)$ to get the equation

$$f(-x) = g(x) - h(x)$$

We now have the 2 equations

$$f(x) = g(x) + h(x) \text{ and } f(-x) = g(x) - h(x)$$

with the 2 unknowns, $g(x)$ and $h(x)$. Adding the equations gives

$$f(x) + f(-x) = 2g(x) \text{ or } g(x) = \frac{f(x) + f(-x)}{2}$$

Substitution gives that

$$g(-x) = \frac{f(x) + f(-x)}{2},\ h(x) = \frac{f(x) - f(-x)}{2},\ h(-x) = \frac{f(-x) - f(x)}{2}$$

We now think we have the correct even and odd functions.

Step 5: Formal Proof: Let f be an arbitrary function with domain D. We will show that there exists an even function g and an odd function h such that

$$\forall\, x \in D,\ f(x) = g(x) + h(x)$$

For every $x \in D,$ let

$$g(x) = \frac{f(x) + f(-x)}{2} \text{ and } h(x) = \frac{f(x) - f(-x)}{2}$$

We now have to show, 1) $f(x) = g(x) + h(x)\ \forall x \in D,$ 2) g is even, and 3) h is odd.

1) Clearly

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \frac{2f(x)}{2} = f(x)$$

2) By substitution,

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(x) + f(-x)}{2} = g(x)$$

Since this is true for every $x \in D,$ g is an even function.

3) By substitution,

$$h(-x) = -\left[\frac{f(-x) - f(-(-x))}{2}\right] = \frac{-f(-x) + f(x)}{2} = \frac{f(x) - f(-x)}{2} = h(x)$$

Since this is true for every $x \in D$, h is an odd function.

When presenting a proof to others, the formal proof in Step 5 is often all that is shown. The proof makes it clear to the audience that the statement is true, but it hides all the thinking that went into its construction. When reading proofs of others, remember that while it may seem simple and obvious, a great deal of thinking and editing went into its discovery.

The proofs in the examples in this chapter have been worked out in more detail than is normally required. The reason for doing this is to help you develop good habits. If you approach the simple problems in an organized manner, then you will find the same habits will help you when the problems become more difficult. Many students avoid these steps in the early chapters to save time when doing homework problems. These students then find they are spending a great deal of time working problems in later chapters, and they are not getting a correct solution after all of this time. A little extra work now will save a great deal of time later, and in fact, many of the later problems will become quite enjoyable to work. In short, **practice good technique on the easy problems.**

§ 3.4: Exercises:

Problem 3.1: Suppose that n is divisible by 3. Then $2n^2 - 3n$ is divisible by 9.

a) Try several values for n, such as 3, 6, and 9 to check that the statement is reasonable.

b) Write what is assumed in usable form.

c) Write what is to be shown in usable form.

d) Prove this claim is true using the Choose Method. In this part, you should both think through the proof and construct a formal proof.

Problem 3.2: Suppose that n is the product of two consecutive even integers. Then n is divisible by 8.

a) Check that the statement is reasonable by checking a few specific cases.

b) Write what is assumed in usable form.

c) Write what is to be shown in usable form.

d) Construct both an informal and a formal proof of claim. What methods did you use?

Problem 3.3: If 3 divides $a^2 + b^2$, then 3 divides a and 3 divides b.

a) Check that this statement is reasonable by computing $a^2 + b^2$ for several different integer values of a and b.

b) Write what is assumed in usable form.

c) Write what is to be shown in usable form.

d) What problems do you have trying to prove this claim using the Choose Method and the forward/backward method?

Problem 3.4: If n and m are consecutive integers, then 4 divides $n^2 + m^2 - 1$.

a) Check this statement for several examples. You might try values where n and m are consecutive positive integers and when they are consecutive negative integers. Does it make a difference?

b) Write what is assumed in usable form.

c) Write what is to be shown in usable form.

d) Prove this result is true. What methods did you use?

Problem 3.5: Consider the statement

$$p \equiv \{\text{if } n \in \mathbb{Z}^+ \text{ divides } m\text{, then there exists a unique } k \text{ such that } m = nk\}$$

a) Check this statement for several appropriate values of n and m.

b) Write what is given, in usable form.

c) Write what is to be shown in usable form.

d) Construct the proof of this statement.

Problem 3.6: If n is a composite number, then $n^2 + 1$ is prime.

a) Write what is assumed in usable form.

b) Write what is to be shown in usable form.

c) Is this claim true or false and why?

Problem 3.7: If n is a power of 2, then $n+8$ is not prime.

a) Write what is assumed in usable form.

b) Write what is to be shown in usable form.

c) Prove this result is true using the Choose Method.

d) Why is the claim "If n is a power of 2, then $n+2$ is not prime" not true?

Problem 3.8: Suppose we have a right triangle, and that c is the length of the hypotenuse and A is its area. Let $p \equiv \{A = c^2/4\}$ and let $q \equiv \{\text{right triangle is isosceles}\}$.

a) Write the statement $s \equiv \{$For a right triangle to be isosceles, it is sufficient that $A = c^2/4\}$ as an implication. Is the implication true and why? Go through the 5 steps.

b) Write the statement $t \equiv \{$For a right triangle to be isosceles, it is necessary that $A = c^2/4\}$ as an implication. Is the implication true and why? Go through the 5 steps.

Problem 3.9: Let $n \in \mathbb{Z}$, c be the length of the hypotenuse of a right triangle, and let a and b be the length of the legs of the triangle. Let $p \equiv \{c^n > a^n + b^n \text{ for all } n > 2\}$.

a) Check the validity of p on the right triangle where $c = 5,\ a = 4,\ b = 3$ using several different values for n. What happens to the inequality as n gets larger? To get an idea of why the inequality holds, you might try the example with $n = 8$. The idea is to rewrite

$$5^8 = 5^2 5^6$$

and use what you know about $5^2$and the Pythagorean theorem. How do you know $5^8 > 4^8 + 3^8$, without using a calculator?

b) Check the validity of p on another right triangle of your choice.

c) Is p true or false, and why? Go through the 5 steps.

Problem 3.10: Let $S = [0, 1]$.

a) Show that if $b = 0$ or $b = 1$, then b is a boundary value for S.

b) Show that if $b \neq 0$ and $b \neq 1$, then b is not a boundary value for S.

Problem 3.11: Let $S = \{0, 1, 2, 3\}$. Show that $b = 1$ is a boundary value for S. Use the 5 steps.

Problem 3.12: Let

$$S = \{\frac{1}{n} : n \in \mathbb{Z}^+\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$$

In this case,

$$S^c = \{x : \forall n \in \mathbb{Z}^+,\ x \neq \frac{1}{n}\}$$

a) Consider the set

$$T = \{x : \exists\, n \in \mathbb{Z}^+ \ni \frac{1}{n+1} < x < \frac{1}{n}\}$$

Show that

$$S^c = T \cup (-\infty, 0] \cup (1, \infty)$$

Remember to show $S^c \subseteq T \cup (-\infty, 0] \cup (1, \infty)$ and $S^c \supseteq T \cup (-\infty, 0] \cup (1, \infty)$

b) Show that $b = \frac{1}{4}$ is a boundary point.

c) Show that $b = 0$ is a boundary point. (You need to clearly write the 5 steps. In step 3, you might need to recall from Glossary that we can assume that for any $x \in \mathbb{R}$, $\exists\, n \in \mathbb{Z}$ $\ni x \leq n$.)

d) Show that if $x \in T$, then x is not a boundary value.

Problem 3.13: If

$$f(x) = \begin{cases} \frac{1}{x} & |x| > 1 \\ x^2 & |x| \leq 1 \end{cases}$$

then f is bounded.

a) What is assumed is already in usable form. Write what is to be shown in usable form.

b) Prove this result is true.

Problem 3.14: If n is odd then n^2 is odd

a) Write what is assumed in usable form.

b) Write what is to be shown in usable form.

c) Prove this result is true.

d) How does the proof of the statement "If n^2 is odd then n is odd" compare with the proof of "If n is odd then n^2 is odd"?

Problem 3.15: If n is the product of 2 odd integers, then n is odd.

a) Write what is assumed in usable form.

b) Write what is to be shown in usable form.

c) Prove this result is true.

Problem 3.16: If n is the sum of an even integer and an odd integer then n is odd

a) Write what is assumed in usable form.

b) Write what is to be shown in usable form.

c) Prove this result is true.

Problem 3.17: If $n \in \mathbb{Z}$ then 3 divides $n^3 - n$.

a) Check that this statement is reasonable by trying several different positive and negative values for n.

b) What is assumed is simple, that $n \in \mathbb{Z}$. Write what is to be shown in usable form.

c) Prove this result is true using the Choose Method. It might help to use cases.

d) Consider the statements $p \equiv \{$If $n \in \mathbb{Z}$ then 3 divides $n^4 - n\}$ and $q \equiv \{$If $n \in \mathbb{Z}$ then 3 divides $n^5 - n\}$. Are either of these statements true or false? Give reasons for you answers.

e) Make conjectures about values of j and n for which $n^j - n$ is divisible by 3.

Problem 3.18: If $i,\ j,$ and k are consecutive integers then 12 does not divide $i^2 + j^2 + k^2 + 1$.

a) Make a table of values for $i^2 + j^2 + k^2 + 1$ when $i = 1,\ 2,\ 3,\ 4,\ 5,$ and 6. In each case, what is the remainder when $i^2 + j^2 + k^2 + 1$ is divided by 12? What pattern do you see?

b) Write what is assumed in usable form.

c) Write what is to be shown in usable form.

d) Prove this claim is true using appropriate cases resulting from observations in part c).

Problem 3.19: Claim: The function

$$f(x) = \begin{cases} 2x+1 & x \leq 2 \\ 3x+2 & x > 2 \end{cases}$$

is increasing on the interval $[0, 5]$.

a) Write what needs to be shown in usable form.

b) Use cases to prove the claim is true.

Problem 3.20: If n is odd and not a multiple of 3 then 24 divides $n^2 - 1$.

a) Make a table of possible n values and resulting $n^2 - 1$ values.

b) Using patterns you observed in table, write what is assumed in usable form (cases). Justify your assumption.

c) Write what is to be shown in usable form.

d) Prove this claim is true using the Choose Method.

Problem 3.21: Consider the function

$$f(n) = \begin{cases} n+4 & n \text{ even} \\ 2n+1 & n \text{ odd} \end{cases}$$

Show that f is one-to-one from $\mathbb{Z}$ into $\mathbb{Z}$. (See Problem 2.23 for definition of **one-to-one**.) Go through the 5 step process.

Problem 3.22: Consider the function

$$f(n) = \begin{cases} 3n-1 & n \text{ even} \\ n+5 & n \text{ odd} \end{cases}$$

Show that f is one-to-one from $\mathbb{Z}$ into $\mathbb{Z}$.

Problem 3.23: Consider the function

$$f(n) = \begin{cases} n+3 & n \text{ even} \\ n-5 & n \text{ odd} \end{cases}$$

Show that f maps $\mathbb{Z}$ onto $\mathbb{Z}$. (See Problem 2.24 for definition of **onto**.) Go through the 5 step process.

Problem 3.24: Consider the function

$$f(n) = \begin{cases} n+6 & n \text{ even} \\ 8-n & n \text{ odd} \end{cases}$$

Show that f maps $\mathbb{Z}$ onto $\mathbb{Z}$. (See Problem 2.24 for definition of **onto**.) Go through the 5 step process.

Problem 3.25: Our goal for this problem is to compare the two numbers

$$0.9999\cdots = 0.\overline{9} \text{ and } 1$$

In parts b) through e), make sure you carefully write what you are given and what you must show is true. This might also consist of constructing a few examples.

a) Show there exists a number between 0.999 and 1.0.

b) Show that $\forall\, \epsilon \in \mathbb{R}^+,\ \exists\, x \in \mathbb{R} \ni 1-\epsilon < x < 1$.

c) Suppose $a, b \in \mathbb{R}$ and $a < b$. Show $r \equiv \{\exists\, x \in \mathbb{R} \ni a < x < b\}$.

d) Suppose $a, b \in \mathbb{R}$ and $a < b$. Let $I = \{x : a < x < b\}$. Let $p \equiv \{\forall\, x_0 \in \text{I},\ \exists\, x_1 \in I \ni x_0 < x_1\}$. Show p is true or construct a counterexample.

e) Suppose $a, b \in \mathbb{R}$ and $a < b$. Let $I = \{x : a \le x \le b\}$. Let $q \equiv \{\forall\, x_0 \in \text{I},\ \exists\, x_1 \in I \ni x_0 < x_1\}$. Show q is true or construct a counterexample.

f) Let $a = 0.999\cdots = 0.\overline{9}$ and let $b = 1.0$. Is r (from part c) true or false in this case? How does this correlate with the result of part c)? Is there a problem?

Problem 3.26: If n is an integer, then 5 divides $n^5 - n$.

a) Check that this statement is true for a few values.

b) What is assumed is clear, $n \in \mathbb{Z}$. What has to be shown?

c) What method seems most likely to work in proving this claim and why?

d) Prove the claim?

Problem 3.27: Let $c \in S \subseteq \mathbb{R}$

Definition: c is an **interior value** of S if and only if there exists an interval (a, b) such that $c \in (a, b) \subseteq S$.

a) Which values $x \in [0, 1] = S$ are interior values and which are not interior values? Prove you claim. Make sure you go through the 5 step process.

b) Let $S = \mathbb{R}/\{1, 2\} = \{x \in \mathbb{R} : x \neq 1, x \neq 2\}$. Which values of S are interior values and which are not interior values? Prove you claim. Make sure you go through the 5 step process.

c) Which values $x \in S = \{x : x = \frac{1}{n}$ for some $n \in \mathbb{Z}^+\}$ are interior values and which are not interior values? Prove you claim. Make sure you go through the 5 step process.

Answers:

3.1a) When $n = 3,\ 2n^2 - 3n = 9$ which is divisible by 9. Similar results hold when $n = 6$ and 9.

3.1b) Assume: $\exists\, k \in \mathbb{Z} \ni n = 3k$.

3.1c) Show: $\exists\, j \in \mathbb{Z} \ni 2n^2 - 3n = 9j$

3.1d) Choose n divisible by 3. Then $\exists\, k \in \mathbb{Z} \ni n = 3k$. Substitution gives

$$2(3k)^2 - 3(3k) = 18k^2 - 9k = 9(2k^2 - k)$$

so $2n^2 - 3n = 9j$ where $j = 2k^2 - k$ □

3.3a) Seems reasonable when checking following Table 3.2

a	b	a^2+b^2	a^2+b^2 divisible by 3?	a and b divisible by 3?
1	1	2	no	no
1	2	5	no	no
1	3	10	no	no
2	2	8	no	no
2	3	13	no	no
3	3	18	yes	yes
3	4	25	no	no
4	4	32	no	no

Table 3.2: Check for when $a^2 + b^2$ is divisible by 3.

3.3b) Assume: $\exists\, k \in \mathbb{Z} \ni a^2 + b^2 = 3k$

3.3c) Show: $\exists\, i, j \in \mathbb{Z} \ni a = 3i$ and $b = 3j$.

3.3d) It is clear that if $a = 3i$ and $b = 3j$, then $a^2 + b^2 = 9k$, but it is not clear how to reverse the process. It seems it would be easier to begin with all the possibilities (cases) that can occur for a and b, that is, $a = 3i,\ a = 3i + 1,\ a = 3i + 2,$ and $b = 3j,$ $b = 3j + 1,\ b = 3j + 2.$ But it still seems as if we are working backwards. Not clear how to rewrite in forward approach.

3.5a) If $n = 5$ and $m = 15$, then we know n divides m. In this case, $k = 3$ is the only integer such that $m = kn$.

3.5b) We are given that $n \in \mathbb{Z}^+$ divides m which means $\exists\, k \in \mathbb{Z} \ni m = nk$. We can assume n m, and k are known numbers.

3.5c) We must show that if $m = jn,$ then $j = k$.

3.5d) If there is an integer $j \in \mathbb{Z}$ such that $m = nj$, dividing by n gives that $j = m/n$. But we are given that $nk = m$, so dividing by n gives that $k = m/n = j$ which is what we needed to show. (Note that since $n \neq 0,$ we know we could divide by n.)

3.7a) Assume: $\exists\, k \in \mathbb{N} \ni n = 2^k$

3.7b) This states that $n + 8$ is composite. By the definition, show $\exists\, i,j \in \{2, 3, \cdots\}$ $\ni n + 8 = ij$

3.7c) Choose $n = 2^k$ where $k \in \mathbb{N}$. Then $n + 8 = 2^k + 8$. There are two cases, $k = 0$ and $k > 0$. Case 1: If $k = 0$, then $n + 8 = 9 = 3 \times 3$, so $i = j = 3$ and $n + 8$ is composite. Case 2: If $k > 0$, then $n + 8 = 2^k + 8 = 2(2^{k-1} + 4)$ so $i = 2,$ $j = 2^{k-1} + 4 \in \{2, 3, \cdots\}$ and $n + 8$ is composite. (since $k > 0$, $k - 1 \geq 0$, so $j = 2^{k-1} + 4$ is an integer greater than 1.)☐

3.7d) In this case, $n + 2 = 2^k + 2 = 3$ when $k = 0$, so $n + 2$ is prime. Claim is not true.

3.9a) Inequality holds. For larger values of n, the left side is even larger than the right side.

$$5^8 = 5^2 5^6 = (3^2 + 4^2)5^6 = 3^2 5^6 + 4^2 5^6 > 3^2 3^6 + 4^2 4^6 = 3^8 + 4^8$$

3.9b) Inequality holds.

3.9c) $c^n = c^{k+2} = c^k(a^2 + b^2) = c^k a^2 + c^k b^2 > a^k a^2 + b^k b^2 = a^n + b^n$

3.11) Step 1: We try some intervals containing $b = 1$. We must show each interval contains a value in S and a value in S^c. For $(x, y) = (0.5, 1.5)$, $1 \in S$ and $1.25 \in S^c$. We found the value in S^c by averaging of b and y. For $(0, 3)$, $1 \in S$ but the average of b and y is 2 which is not in S^c. But in this case, we could try $1.5 \in S^c$. Step 2: We choose (x, y) and assume $x < 1 < y$. Step 3: We must find a such that $x < a < y$ and $a = 0, 1, 2,$ or 3. We must find c such that $x < c < y$ and $c \neq 0, 1, 2, 3$. Step 4: We assume $x < 1 < y$. We let $a = 1$ which is in the interval and in S. Now we must find c. Normally, we would choose

$$c = \frac{1+y}{2} = \frac{1}{2} + \frac{y}{2}$$

since

$$1 = \frac{1}{2} + \frac{1}{2} < \frac{1}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} = y$$

Then $c \in S^c$ unless $c = 0, 1, 2, 3$. Since we know $c > 1$, the only problems are when $c = 2$ or $c = 3$ which occur when $y = 3$ or $y = 5$. But in these cases, we could just let $c = 1.5$. Step 5: Choose $x < 1 < y$ so $1 \in (x, y)$. We must find values a and c in (x, y) such that $a \in S$ and $c \in S^c$. Let $a = 1$. Then

$$a \in (x, y) \cap \{0, 1, 2, 3\}$$

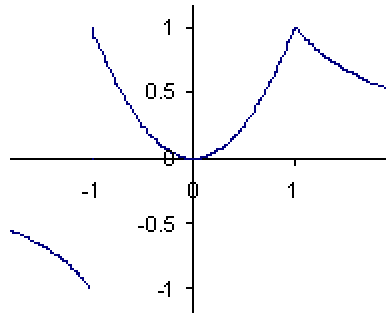
To find c we consider 2 cases. *Case 1:* Suppose $y \neq 3$ and $y \neq 5$. Let $c = (1 + y)/2$. We know that

$$1 < \frac{1+y}{2} < y$$

so $c \in (x, y)$. Since $y \neq 3$ and $y \neq 5$, and $c > 1$, then $c \notin S$ so $c \in S^c$. *Case 2:* Suppose $y = 3$ or $y = 5$. Let $c = 1.5$. Clearly, $c \in S^c$, and since $x < 1 < c < y$, $c \in (x, y)$.

3.13a) Show $\exists\, M \in \mathbb{R}^+$ such that $\forall\, x \in \mathbb{R}$, $|f(x)| \leq M$

3.13b) From the graph below, it appears that $|f(x)| \leq 1$ for all x. So we will let $M = 1$. The problem is that this function is defined differently for different x-values. So we will take cases.



Case 1: Choose $x \in \mathbb{R}$ such that $|x| > 1$. Dividing both sides by $|x|$ gives that $1 > 1/|x| = |f(x)|$. So in this case, $|f(x)| \leq M$.

Case 2: Choose $x \in \mathbb{R}$ such that $|x| \leq 1$. Multiplying both sides by $|x|$ gives that $|x|^2 \leq |x|$, which combined with the first inequality, gives that $|f(x)| = |x|^2 \leq 1 = M$.

So in both cases, $|f(x)| \leq M$ and f is bounded.☐

3.15a) Assume: $\exists\, i, j \in \mathbb{Z} \ni n = (2i+1)(2j+1)$

3.15b) Show: $\exists\, k \in \mathbb{Z} \ni n = 2k+1$

3.15c) We choose $n \in \mathbb{Z} \ni n = (2i+1)(2j+1)$ for some $i, j \in \mathbb{Z}$. Multiplying out gives that $n = 4ij + 2i + 2j + 1 = 2k + 1$ when $k = 2ij + i + j$. Therefore, $n \in \mathbb{O}$.☐

3.17a) It is easy to check that this statement is true for several values, but it is not clear why the statement is true. Let's consider one example in more detail, say when $n = 7$. Then

$$n^3 - n = 7^3 - 7 = 7(7^2 - 1) = 7(7-1)(7+1) = 7(6)(8)$$

We note that $n^3 - n$ simplifies to the product of 3 consecutive integers, one of which must be divisible by 3.

3.17b) Show: $\exists\, j \in \mathbb{Z} \ni n^3 - n = 3j$.

3.17c) Choose $n \in \mathbb{Z}$. We know $\exists\, k \in \mathbb{Z} \ni n = 3k$, $n = 3k+1$, or $n = 3k+2$.

Case 1: Suppose $n = 3k$. Substitution gives $n^3 - n = (3k)^3 - (3k) = 3(9k^3 - k) = 3j$ where $j = 9k^3 - k$.

Case 2: Suppose $n = 3k+1$. Substitution gives $n^3 - n = (3k+1)^3 - (3k+1) = 27k^3 + 27k^2 + 6k = 3(9k^3 + 9k^2 + 2k) = 3j$ where $j = 9k^3 + 9k^2 + 2k$.

Case 3*:* Suppose $n = 3k+2$. Substitution gives $n^3 - n = (3k+2)^3 - (3k+2) = 27k^3 + 54k^2 + 33k + 6 = 3j$ where $j = 9k^3 + 18k^2 + 11k + 2$.

Thus, $n^3 - n$ is divisible by 3.☐

Alternate proof. $n^3 - n = (n-1)(n)(n+1)$. Now, using the assumption

$$\exists\, k \in \mathbb{Z} \ni n = 3k,\ n = 3k+1, \text{ or } n = 3k-1$$

only one substitution needs to be made in each case. In Case 1, $n^3 - n = (n-1)(3k)(n+1)$, in Case 2, $n^3 - n = (3k)(n)(n+1)$, and in Case 3, $n^3 - n = (n-1)(n)(3k)$.)□

3.17d) When $n = 2$, $n^4 - n = 14$ which is not divisible by 3, so p is false. We can factor $n^5 - n = n(n-1)(n+1)(n^2+1)$ and use the same substitutions as in part c) to get that this is divisible by 3, so q is true.

3.17e) It appears that if $n \in \mathbb{R}$ and j is odd, then $n^j - n$ is divisible by 3. If j is even, then $n^j - n$ is divisible by 3 when $n = 3k$ or $n = 3k+1$ for some value of k, but is not divisible by 3 when $n = 3k-1$ for some value of k. See if you can prove these results.

3.19a) Show that if $0 \le x < y \le 5$, then $f(x) < f(y)$.

3.19b) We will consider cases.

Case 1: $0 \le x_1 < x_2 \le 2$. Then $f(x_1) = 2x_1 + 1$ and $f(x_2) = 2x_2 + 1$. Therefore

$$f(x_1) - f(x_2) = 2(x_1 - x_2) < 0 \text{ or } f(x_1) < f(x_2)$$

so condition is satisfied.

Case 2: $2 < x_1 < x_2 \le 5$. This case is similar to Case 1, except we use $f(x) = 3x + 2$.

Case 3: $0 \le x_1 \le 2 < x_2 \le 5$. Then, from cases 1 and 2, $f(x_1) \le f(2) = 5$ and $f(x_2) > 3 \times 2 + 2 = 8$, so $f(x_1) < f(x_2)$.

In all three cases, $f(x_1) < f(x_2)$, so the function is increasing.□

3.21) Step 1: We must show that if $n \ne m$, then $f(n) \ne f(m)$. We can try different pairs of integers and see that they are mapped onto different values. One thing we notice is that even integers are mapped onto even integers and odd integers are mapped onto odd integers. Step 2: We are given the function,

$$f(n) = \begin{cases} n+4 & n \text{ even} \\ 2n+1 & n \text{ odd} \end{cases}$$

and also assume that $n \ne m$. Step 3: We must show that $f(n) \ne f(m)$. Step 4: Let's assume n and m are both odd, that is,

$$n = 2k+1,\ m = 2j+1$$

We want that $f(n) \ne f(m)$, that is

$$\begin{aligned}
&2n+1 \neq 2m+1 \\
&2(2k+1)+1 \neq 2(2j+1)+1 \\
&4k+3 \neq 4j+3 \\
&4k \neq 4j \\
&k \neq j
\end{aligned} \tag{3.3}$$

The cases when n and m are both even, and where one is even and the other odd are similar. Step 5: Assume that $n \neq m$. Case 1: Assume n and m are both odd. Then there exists $k, j \in \mathbb{Z}$ such that $n = 2k+1$ and $m = 2j+1$. Since $n \neq m$, then $2k+1 \neq 2j+1$. Subtracting 1 from both sides gives $2k \neq 2j$. Dividing by 2 gives $k \neq j$. We then have (notice the following is just (3.3) in reverse)

$$\begin{aligned}
&4k \neq 4j \\
&4k+3 \neq 4j+3 \\
&2(2k+1)+1 \neq 2(2j+1)+1 \\
&2n+1 \neq 2m+1 \\
&f(n) \neq f(m)
\end{aligned}$$

which is what we needed to show. Case 2: n and m both even is similar to case 1. Case 3: One integer is even and the other odd. By symmetry, without loss of generality we can assume n is even and m is odd. Then there exists $k, j \in \mathbb{Z}$ such that $n = 2k$ and $m = 2j+1$. (The following are the reverse of the statements we would have written using a backwards proof in step 4 for this case.) Since 1 is odd,

$$1 \neq 2(2j-k) = 4j - 2k$$

Adding $2k+3$ to both sides of the equation gives

$$2k+4 \neq 4j+3 = 2(2j+1)+1$$

or, after substitution,

$$n+4 \neq 2m+1 \text{ or } f(n) \neq f(m)$$

3.23) Step 1: We try picking some integer k, and see if we can find an integer n such that $f(n) = k$. Let $k = 20$. Then we want an n such that either $n+3 = 20$ or $n-5 = 20$. In the first case, $n = 17$, but $f(17) = 17-5 = 12$ since 17 is odd. In the second case, $n = 25$. Checking, we see that $f(25) = 25-5 = 20$ since 25 is odd. Step 2: We are given the function

$$f(n) = \begin{cases} n+3 & n \text{ even} \\ n-5 & n \text{ odd} \end{cases}$$

We are also given an $f(n)$-value, k. The value k is treated as a known number. Step 3: We must find an n such that $f(n) = k$. The value n is unknown. Step 4: In checking some values, it seem as if finding n depends on whether k is even or odd. So we do cases. Case 1: Assume k is odd, that is, there exists $j \in \mathbb{Z}$ such that $k = 2j+1$. We know $f(n) = n + 3$ or $n - 5$. We try each. If

$$n + 3 = 2j + 1, \text{ then } n = 2j - 2 = 2(j - 1)$$

On the other hand, if

$$n - 5 = 2j + 1, \text{ then } n = 2(j + 3)$$

In either case, n is even. But if n is even, then $f(n) = n + 3$. So if k is odd, we let $n = 2(j - 1)$. (Note that $2(j - 1) = k - 3$ since $k = 2j + 1$.) Similarly, we find that if k is even, that is, there exists $j \in \mathbb{Z}$ such that $k = 2j$, then $n = k + 5$. Step 5: Choose $k \in \mathbb{Z}$. Let

$$n = \begin{cases} k - 3 & k \text{ odd} \\ k + 5 & k \text{ even} \end{cases}$$

If k is odd, then $k - 3$ is even, so

$$f(n) = n + 3 = (k - 3) + 3 = k$$

If k is even, then $k + 5$ is odd, so

$$f(n) = n - 5 = (k + 5) - 5 = 5.$$

So for every $k \in \mathbb{Z}$, there exists an $n \in \mathbb{Z}$ such that $f(n) = k$, so f is onto $\mathbb{Z}$.